DIPLOMA THESIS

BROUWER'S FAN THEOREM

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Preface

In 1925 A. Ekonomou, a professor then of Athens Polytechnic School, made the first report of Brouwer's program in the Greek mathematical literature in a lecture before the Greek mathematical society[1]. This was also the last in the 20's, a decade in which the community of central European mathematics faced the most fierce debate of contemporary mathematics, the debate on the foundations of mathematics. Since its protagonists were some of the most important German mathematicians of that period (such as Hilbert and Weyl) and the German speaking Dutch mathematician Brouwer, and the most influential journal was Mathematische Annalen, the term that sealed that period is the German word *Grundlagenstreit*.

Hilbert had organized the Göttingen circle, gathering around him important mathematicians, like Ackermann, von Neumann, Bernays, occasionally philosophers, like Husserl and Nelson, and physicists, like Born. In the foundations of mathematics he was the leading personality of his program, *formalism*, while in physics he pursued a program of axiomatic foundation of physical theories and of solid mathematical reconstruction of them.

Brouwer, after an explosive topological period, almost devoted himself to his foundational program, *intuitionism*, which, mainly through his direct followers (like Heyting), rather then himself, was transformed into a school of reconstruction the whole of mathematics. Through the work of Troelstra and van Dalen, the leading names of the next generation, this school is still alive in Holland.

Weyl moved philosophically between both, Hilbert and Brouwer. In 1918 though, independently from them, with strong philosophical motives (Husserl and Fichte were two serious influences for him), Weyl contributed on the foundations of mathematical analysis with his original work, "Das Kontinuum", the origin of *predicativism*. In the early 20's he espoused Brouwer's intuitionism, disappointing his former Göttingen teacher, Hilbert. Later he expressed his doubts on Brouwer's program, accepting a kind of Hilbert's prevail, recognizing though, the closeness of the two programs. He never stopped admiring Brouwer and stressing the value of intuitionism.

The echoes of **Poincaré**'s foundational views, and the rebirth of **Frege**'s *logicism* by **Russell**, which had influenced even Hilbert between his early and later foundational period, complete the philosophical scenery of that period. Finally, in the early 30's **Gödel**'s incompleteness theorems, the philosophical value of which is still discussed, determined not only modern mathematical logic, but also the outcome of the foundational debate.

Philosophers of science, like members of the **Vienna circle**, mathematicians, like **Ramsey**, philosophers, like **Becker** and **Wittgenstein**, were seriously involved, or influenced by the debate.

The Grundlagenstreit was shaped by the personalities of the two of the most important mathematicians of that period, Brouwer and Hilbert. Also, it was the product of conceptual changes in the mathematics, starting in the 17th and culminated in the 19th century, and an expression of the close connection between mathematical and philosophical thought in the mind of most of the great mathematicians previously mentioned.

All of them were well aware of the work of the most influential personality in the phi-

[1] [Ekonomou 1926] p.80.

losophy of mathematics, **Kant**. More or less, all foundational programs of that period were different responses to the Kantian model, after the mathematical "revolutions" of the 19th century and the "revolutions" in physics of the early 20th century.
Even if Grundlagenstreit often had a polemic character, it was an expression of the high level of the interconnection between mathematical and philosophical thought of that period. Unfortunately, the "winner" of the debate was the weakest opponent, *naive mathematical realism*, the kind of mathematical realism which "justifies" all of standard set-theoretical mathematics. While both, mature formalism and intuitionism (early and mature), agreed on their critique on mathematical realism, it was this "poor" foundational framework which suited better to post-war mathematical community.

It was the conceptual changes brought by the development of non-Euclidean geometries, the degeometrization and arithmetization of analysis, which influenced the main foundational programs, rather than the set-theoretical paradoxes, as it has often been said[2]. Although the resolution of paradoxes was an important issue, it was more important to deal with the conceptual problems that were responsible for those paradoxes.
The main objective of our study is to present Brouwer's Fan theorem (BFT), the most central theorem of Brouwer's Intuitionistic Analysis (BIA).
BIA is presented here in the spirit of Brouwer, meaning that we try to preserve Brouwer's non set-theoretic mentality, which is often neglected in modern presentations of intuitionism. Another non-standard element of our presentation is that intuitionistic logic is not considered a beginning point of BIA, rather a necessary aftermath, in accordance to Brouwer's thought. His conclusions on the Principle of Excluded Middle (PEM) follow his basic assumption on the nature of the fundamental objects of BIA. Brouwer himself left the formalization of intuitionistic logic to his pupil Heyting.
On the whole, although post-Brouwer intuitionism turned out as a legitimate branch of formal mathematics, it lost Brouwer's revolutionary spirit. That was a result of certain "social" conditions and also of weaknesses of BIA itself[3].
Only a few mathematicians nowadays believe that there is a crisis in the foundations of mathematics. Weyl's views ([Weyl 1946] p.13) echo a very distant past.

> [This history should make one thing clear: we are less certain than ever about the ultimate foundations of (logic and) mathematics; like everybody and everything in the world today, we have our crisis. We have had it for nearly fifty years. Outwardly it does not seem to hamper our daily work, and yet I for one confess that it has had a considerable practical influence on my mathematical life: it directed my interests to fields I considered relatively safe, and it has been a constant drain on my enthusiasm and determination with which I pursued my research work. The experience is probably shared by other mathematicians who are not indifferent to what their scientific endeavours mean in the contexts of mans whole caring and knowing, suffering and creative existence in the world.]

More polemic was **Bishop**'s isolated viewpoint[4].

> [There is a crisis in contemporary mathematics, and anybody who has not noticed it is being willfully blind. *The crisis is due to our neglect of philo-*

[2]See e.g., [Giaquinto 2002].
[3]See [Heyting 1962].
[4]See [Bishop 1975] p.507.

> *sophical issues.* The courses in the foundations of mathematics as taught in our universities emphasize the mathematical analysis of formal systems, at the expense of philosophical substance. Thus it is the mathematical profession that tends to equate philosophy with the study of formal systems, which require knowledge of technical theorems for comprehension. They do not want to learn yet another branch of mathematics and therefore leave the philosophy to the experts. As a consequence, we prove these theorems and we do not know what they mean. The job of proving theorems is not impeded by inconvenient inquires into their meaning or purpose. In order to resolve one aspect of this crisis, emphasis will have to be transferred from the mechanics of the assembly line which keeps grinding out the theorems, to an examination of what is being proved.]

In our view, in modern times "organization" has replaced foundation. The questions that bothered Brouwer's generation still need to be examined, despite the prevailing set-theoretical framework. The nature of continuum, the relation between language and mathematics, maybe need to be reinvestigated.

The main purpose of our thesis is to present and discuss Brouwer's proof of BFT. In order to give a self-contained exposition of fan theorem an introduction to Brouwer's basic concepts and foundational principles is also given. Brouwer developed from the beginning an interpreted mathematical theory of the continuum. Mathematical continuum for him is the mathematical expression of a certain intuition. The realization of this intuition belongs though, to his mature period, through the concept of spread. This foundational attitude of Brouwer has many important consequences that post-Brouwer intuitionism completely neglects.
Brouwer, trying to avoid the use of absolute infinity in the mathematical treatment of the continuum, introduced choice sequences generated by the spread law. These are "incomplete" objects or "on-going" objects and they are responsible for Brouwer's "new logic". All Brouwer's deviations from classical mathematics result from his use of new concepts and his constructive methods. The concept of intuitionistic function depends on Brouwer's new concept of the continuum and since intuitionistic functions are crucial to the formulation of fan theorem their study is included.
The proof of fan theorem is based on [Brouwer 1927]. Fan theorem is a consequence of bar theorem, the proof of which is highly non-standard in structure, even for today standards.
We also examine some consequences of fan theorem and especially Brouwer's Uniform Continuity theorem (UCT), according to which, a function simply defined on $[0,1]$ is uniformly continuous. Just like continuity principle and fan theorem, this fact is the result of the study of a different kind of function, "intuitionistic Function".
Brouwer himself was never fully satisfied with his proof of fan theorem, although he considered it an intuitionistic truth. We analyze some of the contemporary critique on his proof and we provide another *intuitionistic* argument against the intuitionistic validity of Brouwer's proof, independently found by us.
We also give, as an Appendix, a classical development of the basic facts about Baire and Cantor space. In that way we present the classical behavior of those classical objects which have an intuitionistic analogue. We believe that all these classical results are in some sense necessary to the understanding of the differences between classical and

intuitionistic analysis.
We refer to pages of Brouwer's papers through [Brouwer 1975] collective volume.

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Contents

1. Introduction to Brouwer: thesis, topology and intuitionism. The Dutch mathematician Luitzen Egbertus Jan Brouwer (Overschie 1881- Blaricum 1966) was one of the greatest mathematicians of his time[5]. Dieudonné, in [Dieudonné 1989], p.161], refers to Brouwer's epoch-making results of 1910-1912 as the "*first proofs* in algebraic topology, since Poincaré's papers can only be considered as blueprints for theorems to come".
Brouwer's topological theorems gave Brouwer international fame and recognition. Their proofs were based on new concepts and methods of his. Throughout his mathematical life Brouwer was a great creator of concepts. In 1910 he defined rigorously the concept of degree of a continuous map and relying exclusively on it he proved, mostly through "fantastically complicated constructions" the celebrated Brouwer's theorems[6].
At the beginning of his great topological period Brouwer, tackling Hilbert's 5th problem[7], showed that all C^0 groups of transformations of the real line are in fact Lie groups. Attempting to extend this result to transformation groups of $\mathbb{R}^2$, he studied the then known topology of the plane. Studying the related to this subject papers of Schönflies, he discovered many counterexamples to Schönflies' results. The most unexpected of these brought him immediate recognition. Brouwer constructed a compact, connected subset of the plane, which cannot be written as the union of two proper compact, connected subsets and it is the frontier of three connected components of its complement. Dieudonné describes accurately what followed (in [Dieudonné 1989], pp.168-9):

> [From these early papers it would have been difficult to foresee the breakthrough accomplished by Brouwer in the years 1910-1912, owing to a complete change of outlook and a remarkably skillful use of the new concept *simplicial approximation* that he introduced... In a rapid succession of papers published in less than two years, the "Brouwer's theorems" (as they are still called) made him famous overnight. They solved a whole batch of problems on n-dimensional spaces for *arbitrary* n that had looked intractable to the previous generation: invariance of dimension of open sets in $\mathbb{R}^n$, invariance of domain, extension of the Jordan curve theorem, existence of fixed points of continuous mappings, singularities of vector fields, and finally, based on ideas of Poincaré and Lebesgue, a definition of the notion of *dimension* for arbitrary compact metric spaces.
> In retrospect, it therefore seems legitimate to consider Brouwer as the cofounder, with Poincaré, of simplicial topology. More precisely, it may be said that Poincaré defined the *objects* of that discipline, but it is Brouwer who imagined *methods* by which theorems about these objects could be *proved*, something Poincaré had been unable to do.
> ... two features that characterize almost all his proofs of 1911-1912: a remarkable originality and a great complexity.]

We give here an extremely brief account of "Brouwer's theorems" and new concepts:

(a) The invariance of dimension: There is no homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^m$, if $n \neq m$.

[5] [van Dalen 1999] and [van Dalen 2005] comprise his complete biography.
[6] Most of them are now simple consequences of homology theory.
[7] Hilbert's 5th problem: Lie's concept of a continuous group of transformations without the assumption of the differentiability of the functions defining the group.

Cantor, in 1877, had discovered a bijection of $\mathbb{R}$ onto $\mathbb{R}^n$, a complete surprise that seemed to threaten the foundations of analysis[8]. Peano's curve (1890) was an example of a continuous (but not injective) map of $\mathbb{R}$ onto $\mathbb{R}^n$[9]. In order to save the intuitive concept of dimension, Dedekind soon conjectured that there is no bicontinuous bijections (homeomorphisms) of $\mathbb{R}^n$ onto $\mathbb{R}^m$ for $m \neq n$. It was Brouwer who proved Dedekind's conjecture, securing (ironically) the foundations of classical analysis.

(b) The invariance of Domain: If $f : U \to \mathbb{R}^n$ an $1-1$, continuous function, where U is an open subset of $\mathbb{R}^n$, then $f(U)$ is open in $\mathbb{R}^n$.

Hence, the property of being a domain of $\mathbb{R}^n$ (i.e., a connected, open subset of $\mathbb{R}^n$) is invariant under $1-1$, continuous functions on U and in $\mathbb{R}^n$.

The invariance of domain implies the invariance of dimension: Let $n > m$ and $e : \mathbb{R}^n \to \mathbb{R}^m$ a supposed homeomorphism. Then, if $h : \mathbb{R}^m \to \mathbb{R}^n$ is the $1-1$ and continuous mapping

$$(x_1, x_2, ..., x_m) \mapsto (x_1, x_2, ..., x_m, \underbrace{0, 0, ..., 0}_{n-m}),$$

the composite function $h \circ e$ is a $1-1$ and continuous function of $\mathbb{R}^n$ into $\mathbb{R}^n$, therefore $(h \circ e)(\mathbb{R}^n)$ should be open, which is not, since there is no ball of $\mathbb{R}^n$ around $(0, 0, ..., 0)$ contained in $(h \circ e)(\mathbb{R}^n)$.

(c) The Jordan-Brouwer theorem: If Σ is a subset of $\mathbb{R}^n$ homeomorphic to the sphere S_{n-1}, then $\mathbb{R}^n$-Σ has exactly two connected components.

This is a generalization of Jordan's curve theorem (that a closed plane curve separates the plane in two parts: its bounded interior and its unbounded exterior).

(d) The no separation theorem: If U is a connected open subset of $\mathbb{R}^n$, and $F \subset U$ is a homeomorphic image of a compact subset K of S_{n-1}, distinct from S_{n-1}, then $U - F$ is connected.

(e) The fixed-point theorem: If $f : D_n \to D_n$ is a continuous function of the closed ball D_n of $\mathbb{R}^n$ to itself, then f has a fixed point i.e., there is an x in D_n such that $f(x) = x$.

The fixed point theorem is equivalent to the fact that the identity map $I : S^n \to S^n$ is not null-homotopic i.e., it is not homotopic to a constant map. It is no surprise that Brouwer was the first who gave the definition of homotopy in 1911[10].

Definition of dimension of a compact metric space: As Dieudonné points out "the theorem of invariance of dimension did not give a definition of the word "dimension" as a number attached to a topological space and invariant under homeomorphisms except for spaces locally homeomorphic to $\mathbb{R}^n$, and even for these spaces the introduction of the auxiliary space $\mathbb{R}^n$ was not satisfactory for a notion that should have been an intrinsic one". Brouwer, relying on ideas of Poincaré and Lebesgue, defined a space of dimension 0 as one containing no connected set with more than one point, and a space X of dimension $n > 0$ by the property that n is the smallest positive integer such that any two disjoint closed subsets of X are separated by a subset of dimension $\leq n-1$. X has dimension n at a point P, if P has a fundamental system of neighborhoods of

[8] Such a bijection can be found e.g., in [Enderton 1977] pp.148-49.

[9] Peano's curve is presented e.g., in [Gelbaum-Olmsted 1964] pp.133-34.

[10] For a modern reconstruction of Brouwer's theorems see e.e.g., [Dugundji 1989] and for a comprehensive presentation of Brouwer's original results and concepts see [Dieudonné 1989].

dimension n. Brouwer then proved that with his definition $\mathbb{R}^n$ has dimension n at every point.

The above results form the core of Brouwer's topological period[11]. Brouwer did not publish any important paper on topology after 1913, devoting his efforts to an intuitionist reconstruction of mathematics. Doing so, e.g., Hirsch notes (in [Hirsch 1976] p.141) that Brouwer "repudiated some of his earlier results". We briefly explain why this was unavoidable.

Brouwer's 1911-proofs of fixed point theorem were non-constructive from his intuitionistic point of view. Brouwer, in the paper of 1911 in which he gave the definition of the degree of a map, realized that this notion could be used to prove that a continuous map $f : S^n \to S^n$, such that $deg(f) \neq (-1)^{n+1}$, has at least one fixed point. What he showed was that if f has no fixed point, then $deg(f) = (-1)^{n+1}$. As we explain in Paragraph 1.3, this is not a purely constructive existence proof, since the fixed point possesses only a "logical" and not a "real" existence, at least within a certain constructive tradition. Modern proofs are also non-constructive, reductio ad absurdum proofs. In [Brouwer 1952A] though, Brouwer not only gives a concrete intuitionistic proof that the fixed-point theorem on the sphere in its classical form does not hold intuitionistically, but also proves an intuitionistic fixed-point (core) theorem.

From the beginning Brouwer's attitude towards his topological theorems was connected to his philosophical ideas. Brouwer himself mentions (see [van Dalen 1999], p.178 or [Brouwer 1928a]):

> [I have restricted myself to the laying of the foundations of the theory of dimension, and refrained from further dimension theoretic developments, on the one hand because with the proof of the justification theorem[12] the intended purpose had been reached, on the other hand because an intuitionistic realization of the subsequent considerations ... was, in contrast to the justification theorem, not plausible.]

Koetsier and van Mill (in [Koetsier, van Mill 1997] p.145) state that:

> [Brouwer's work in dimension theory is constructive in the sense of the first part of his dissertation ... manifolds are constructed out of simplexes and manifolds and continuous mappings are handled by means of potentially

[11] There are many other results connected to Brouwer such as:

(i) The Phragmén - Brouwer theorem: If K is a compact connected subset of $\mathbb{R}^2$, then the boundary of each connected component of $\mathbb{R}^2$-K is a connected subset.

(ii) The Poincaré - Brouwer theorem: Every continuous non-vanishing vector field on an even-dimensional S^{2n} must contain at least one normal vector. In particular, there can be no continuous non-vanishing vector field of tangential directions on any S^{2n}.
As a consequence, each $f : S^{2n} \to S^{2n}$ either has a fixed point or sends a point to its antipode.

(iii) Brouwer's reduction theorem: If F is a closed subset of of a second countable topological space X and F possesses an inductive property P, there is an irreducible closed subset of F which possesses P.

A property P of subsets of X is called inductive iff whenever each member of a countable nest of closed sets has P, then the intersection has P. Also a set F is irreducible with respect to P iff no proper closed subset of F has P.

but we mention here only the most famous.

[12] That $\mathbb{R}^n$ has dimension n.

infinite systems of approximations similar to the way in which in the dissertation the continuum is handled by means of the dual scale[13]...his topological notions always refer to systems that can be considered as mentally constructed...the fact that there are instances in his topological work where Brouwer sins against his own intuitionistic views, does not run counter to the existence of a basic unity between the work in his dissertation and his topological work.]

Brouwer showed the same kind of mathematical power and originality in his own program of foundation and reconstruction of mathematics, intuitionism, on which he almost devoted his creative powers. Although the intuitionism of French mathematicians Borel, Lebesgue and Poincaré shares some common philosophical ideas with Brouwer's intuitionism, it did not consist an organized program of foundation of mathematics with a technical influence on Brouwer.

Brouwer's intuitionism may be divided in three periods: early intuitionism (1907-1915/6), mature intuitionism (1915/6-1927/8) and late intuitionism (1927/8-1955).

Early intuitionism (1907-1915/6): Starting point of this period is Brouwer's doctoral thesis of 1907 "Over de Grondslagen der Wiskunde"[14]. Besides some purely mathematical contributions, like Brouwer's partial solution to Hilbert's 5th problem, his thesis is a systematic presentation of the original philosophical views of young Brouwer on the foundations of mathematics and his critical comments on the prevailing views of that era.

In his thesis we find for the first time the now standard distinction between mathematics (mathematics of the first order) and meta-mathematics (mathematics of higher order), a distinction made by Brouwer in order to give an elegant critique on Hilbert's early formalism ([Brouwer 1907], pp.194-195). Brouwer rightly demanded priority in the mathematics - meta-mathematics distinction from Hilbert, in [Brouwer 1928].

As van Stigt mentions, (in [van Stigt 1990], p.viii), Brouwer's thesis "was the manifesto of an angry young man taking on the mathematical establisment on all fronts"[15]. This "anger" though is not only a result of youth, that would undermine Brouwer's argumentation. For Brouwer mathematical truth goes beyond mathematics, reflecting human mind itself. As Brouwer notes, in [Brouwer 1981] p.90:

> [The stock of mathematical entities is a real thing, for each person, and for humanity.]

On the whole, Brouwer's thesis and the papers that followed it in the early period form an exposition of general philosophical principles but not a reconstruction of mathematics based on these principles.

Mature intuitionism (1915/6-1927/8): It is the reconstruction of mathematics based, roughly, on the fundamental principles of early intuitionism. The mathematical continuum is the main object of study. Intuitionistic analysis is the mathematical study

[13] The dual scale is a potentially infinite systems of points (cuts) which Brouwer applies on the preexisted continuum.

[14] Translated as "On the Foundations of Mathematics" in Brouwer's collected works.

[15] A complete, mostly historical, description of Brouwer's thesis, which was reedited by van Dalen ([van Dalen 2001]), is [Kuiper 2004].

of the continuum as it is interpreted through the concept of spread. Although continuum was in early intuitionism as fundamental as the concept of number, in mature intuitionism it is described through a generator of sequences, the spread of reals (see Paragraph 5).
Mature intuitionism starts in 1915/6 where the continuity principle is found in Brouwer's lectures, a starting point of Brouwer's analysis, and together with all other Brouwer's concepts and results lead to Brouwer's fan theorem and Uniform Continuity theorem. In 1927 Brouwer gives the most standard proof of fan theorem, the one we describe in Paragraph 11, which puzzled his contemporaries and it is still a matter of debate, with respect to its compatibility with the rest of intuitionistic epistemic principles. Brouwer's period of reconstruction of mathematics ends, roughly, a year later with the "Annalen affair".

Late intuitionism (1927/8-1955): Brouwer did not provide a significantly new theorem or a new proof of fan theorem in this period of his life. His late contributions are extensions or recapitulations of his mature period. 1955 is the year of his last contribution on intuitionism.

In our opinion, Brouwer was the greatest *philosopher-mathematician* of his era. That is, regarding the foundations of mathematics he reacted more than a philosopher rather than a mathematician, by not hesitating to deny a large part of his (even own) contemporary mathematics in order to be consistent with his philosophical beliefs. In contrast, Russell and Hilbert were the great *mathematicians-philosophers* of their time. That is, their philosophical ideas were technically influenced by their need to secure the whole of their contemporary mathematics. Russell's Axiom of Reducibility and late Hilbert's program exemplify this.
One of the main reasons that Brouwer's intuitionism was treated as a curiosity to be dismissed by his contemporaries was its exclusion of large parts of accepted set-theoretical mathematics and its inclusion of results which contradict classical mathematics. Brouwer's views, developed in a milieu favoring abstraction against constructivism, often had a polemic character. But his passion was the result of the struggle of a honest thinker and great mathematician to express his original ideas.
Brouwer was often seen as a curiosity himself, as a person "eager to contradict"[16]. Smorynski (in [Smorynski 1977] p.822) talks about Brouwer's "bizarre attempt to turn mathematics into a religion" and "when, in 1920, Weyl fell prey to Brouwer's lunacy, David Hilbert decided to intervene". These expressions seem to us, to say the least, completely unjust.
In 1912 Brouwer became a professor at the University of Amsterdam. His inaugural address "Intuitionism and Formalism" was a severe attack on the extremities of early formalism and the axiomatic method and a call for the transformation of mathematics along the intuitionistic principles. The debate between intuitionism and formalism turned into a debate between Brouwer and Hilbert. In 1928 Brouwer was excluded, on Hilbert's decision, by the editorial board of Mathematische Annalen, the most important mathematical journal of that period[17]. Brouwer collapsed and withdrew from the Grundlagenstreit two years before Hilbert's program receive a serious blow, Gödel's

[16]These words, heard in Logic Colloquium talk in Athens 2005, made, unfortunately, many people to laugh.

[17]For the details of this sad story see [van Dalen 2005] and [Reid 1986] pp.184-188.

proof of unprovability of the consistency of arithmetic[18].
Brouwer was a powerful mathematical and philosophical mind whose impact can be found in all aspects of modern constructivism, modern logic and modern topology. In the 20th century two schools of research flourished in Holland, intuitionistic mathematics and classical topology, both stemming from Brouwer's work. His general philosophical ideas were a major contribution to the local Dutch philosophical movement "Significs", while his philosophical 1929 work "Mathematik, Wissenschaft und Sprache" had an impact on Gödel regarding the relation between language and mathematics[19].

2. Trees, fans and König's lemma. We shall give here the basic definitions and facts of a "language" basic for the rest of the technical part of our study. All concepts of this paragraph are understood classically.
Let X be a non-empty set and $n \in \mathbb{N}$. X^n is the set of finite sequences of length n of elements of X. I.e.,

$$X^n = \begin{cases} \{u : \{0, 1, 2, ..., n-1\} \to X\} & \text{, if } n \neq 0 \\ \emptyset & \text{, if } n = 0 \end{cases}$$

where $\emptyset$ denotes here the empty sequence. The set of all finite sequences of elements of X is denoted by $X^{<\mathbb{N}}$ and

$$X^{<\mathbb{N}} = \bigcup_{n \in \mathbb{N}} X^n.$$

The length of a finite sequence u of elements of X is denoted by $l(u)$, while $l(\emptyset) = 0$. If $l(u) = n$, we say that u is an n-sequence. If $u = (u_0, u_1, ..., u_{n-1})$ and $w = (w_0, w_1, ..., w_{k-1})$ belong to $X^{<\mathbb{N}}$, then u *is an initial segment of* w, $u \preceq w$, iff $n \leq k$ and $u = w|_k$. Then we say that w *dominates* u or w is a *descendant*, or an *extension* of u and u is an *ancestor* of w. Also, u, w are said *compatible* iff $u \preceq w$ or $w \preceq u$. Otherwise, they are called *incompatible* and we denote this by $u \bowtie w$.
The *concatenation* $u \frown w$ of u, w is the finite sequence

$$(u_0, u_1, ..., u_{n-1}, w_0, w_1, ..., w_{k-1}).$$

The concatenation $u \frown (k)$ of finite sequence u with the 1-sequence (k) is denoted $u \frown k$ and $u \frown k$ is called an *immediate successor* of u, or u an immediate predecessor of $u \frown k$.
$X^{\mathbb{N}}$ is the set of all infinite sequences α[20] of elements of X i.e.,

$$X^{\mathbb{N}} = \{\alpha | \alpha : \mathbb{N} \to X\}.$$

A finite sequence u *is an initial segment of* α, $u \prec \alpha$, iff there is n such that $u = \alpha|_n$. We also write then, $\alpha|_n = n_\alpha$.
A **tree on** X is a subset $T \subseteq X^{<\mathbb{N}}$ closed under initial segments i.e.,

$$(w \in T \ \wedge \ u \prec w) \Rightarrow u \in T.$$

[18]The proof of the consistency of arithmetic was vital to the completion of Hilbert's formalism. Even from 1900 the consistency of arithmetic was second in Hilbert's list. In 1899 Hilbert had reduced the consistency of geometry to the consistency of arithmetic.

[19]For this influence see [Hesseling 2003] pp.281-86.

[20]Throughout this work we use small Latin letters for finite sequences and Greek small letters for infinite sequences.

If there is $w \in T$, i.e., if T is non-empty, then $\emptyset$ always belongs to T, since $\emptyset \preceq w$. Each element of T is called a *node* (or *branch*, or *path*) of T, while an *infinite branch* of T is a sequence α of $X^{\mathbb{N}}$ such that $n_\alpha \in T$ for each n. The body, $[T]$, of T is the set of all infinite branches of T, i.e.,

$$[T] = \{\alpha | \alpha \in X^{\mathbb{N}} : (\forall n)(n_\alpha \in T)\}.$$

A tree T is called *pruned* iff every u in T has a proper extension $w \succ u$. Also, a tree T is called *splitting* iff each node u of T has incompatible extension nodes in T i.e.,

$$(\forall u \in T)(\exists w_1, w_2 \in F)\ u \prec w_1, w_2 \quad \wedge \quad w_1 \bowtie w_2.$$

We may correspond to a tree (X, T) a structure (A, l, S) where A is a non-empty set of *points*, $l : A \to \mathbb{N}$, with $l(a) = n$ is the level of point a, and the relation aSb, "b is an immediate successor of a" satisfies the following:

s_1: $\exists! a_0$ such that $l(a_0) = 1$, and a_0 is called the root of the tree.
s_2: $\forall b \neq a_0\ \exists! a$: aSb, i.e., each point besides the root has a unique predecessor.
s_3: If aSb, then $l(b) = l(a) + 1$

Obviously, the correspondence is established by interpreting the nodes of T as points, $(A = T)$, the $\emptyset$ sequence as the root of the tree, which we also denote by $<>$, the length of a node plus 1 as the level of a node and the uSw relation as the immediate extension relation of a node u of T. Then, a tree clearly has the well-known tree-visualization.
A subtree S of a tree T, $S \preceq T$[21], is a tree such that $S \subseteq T$.
Characteristic examples of trees on X are the Baire tree $\mathcal{X}$, where $\mathcal{X} = X^{<\mathbb{N}}$ and the Cantor trees $\mathcal{C}_X$, where $\mathcal{C}_X = \{x_0, x_1\}^{<\mathbb{N}}$ and x_0, x_1 belong to X. Obviously, $\mathcal{C}_X \prec \mathcal{X}$.
If $A \subseteq X^{\mathbb{N}}$ and T is a tree on X, we define the set A^* of initial segments of elements of A which *cut* the tree T, i.e.,

$$A^* = \{n_\alpha | \alpha \in A\ \wedge\ n_\alpha \in T\} \cup \{\emptyset\}.$$

Proposition 2.1: (i) $A^* \preceq T$.
(ii) $A \cap [T] \subseteq [A^*]$.
(iii) $[T]^* = T$ and $A = [T] \Rightarrow [A^*] = A$.

Proof: (i) Let $w \in A^*$ and $u \preceq w$. By definition $w = n_\alpha$ for some n and some α in A. So, $u = m_\alpha$ for some $m \leq n$, therefore $u \in A^*$. Since $\emptyset$ belongs also to A^*, A^* is closed under initial segments and since $A^* \subseteq T$, $A^* \preceq T$.
(ii) If $A \cap [T] = \emptyset$, then (ii) holds trivially. If there is $\alpha \in A \cap [T]$, then $\alpha \in [T]$, hence n_α is in T for each n. Therefore, $\alpha \in [A^*]$.
(iii) $[T]^* = T$ is trivial and using it we get $[A^*] = A$ if $A = [T]$.$\diamond$

In order to give an example of an A such that $A \cap [T] \subsetneq [A^*]$, consider $A = X^{\mathbb{N}} - \{\alpha\}$, where α any element of $X^{\mathbb{N}}$ and the Baire tree $\mathcal{X}$. Each n_α is an n_β for some β in $X^{\mathbb{N}}$, therefore, $[A^*] = [\mathcal{X}] = X^{\mathbb{N}} \supsetneq A$.
$X^{\mathbb{N}}$ becomes a topological space as the $\mathbb{N}$-product of X with the discrete metric. The standard basis for its topology is the family of the sets

$$B(u) = \{\alpha | \alpha \in X^{\mathbb{N}}\ :\ u \prec \alpha\},$$

[21] We use for simplicity the same symbol of partial relation $\preceq$ while its context is made clear by the fixed use of symbols of objects.

satisfying:

(i) $u \prec w \Rightarrow B(w) \subseteq B(u)$.
(ii) $u \bowtie w \Rightarrow B(w) \cap B(u) = \emptyset$.
(iii) $\bigcup_u B(u) = X^{\mathbb{N}}$ and, if $t \prec u$ and $t \prec w$, then $B_t \subseteq B(u) \cap B(w)$.
$X^{\mathbb{N}}$ is also a metrizable space and there is a bijection between pruned trees on X and closed subsets of $X^{\mathbb{N}}$[22].

If T is a tree on X and S is a tree on Y, a map $\Phi^* : T \to S$ is called *monotone* iff

$$u \preceq w \Rightarrow \Phi^*(u) \preceq \Phi^*(w).$$

A monotone function Φ^* is extended to a function $\Phi : D(\Phi^*) \to [S]$, where

$$D(\Phi^*) = \{\alpha | \alpha \in [T] \ : \ lim_n l(\Phi^*(n_\alpha)) = \infty\},$$

and Φ is defined by

$$\Phi(\alpha) = \bigcup_n \Phi^*(n_\alpha).$$

If $D(\Phi^*) = [T]$, then Φ^* is extendable to the whole body of T and it is called *proper*.
Note that if $\Theta(\alpha) = \beta$, where Θ is a function on infinite branches of a tree, classically there is no need to know how this correspondence has become possible. The concept of classical function is too abstract, even the argument and its image are infinite objects. These too are considered known or given and no question arises on how we know them. So, if w is an initial segment of β, then by continuity of Θ, $\alpha \in B(u) \subseteq \Theta^{-1}(B(w))$, for some u i.e.,

$$(*) \qquad (\forall w \prec \beta)(\exists u \prec \alpha)(\gamma \succ u \Rightarrow \Theta(\gamma) \succ w).$$

Next result shows that continuity of Θ is connected to a mechanism which "explains" how $\Theta(\alpha) = \beta$ is possible.

Proposition 2.2: (i) Let $\Phi^* : T \to S$ a monotone map. Then, $D(\Phi^*)$ is G_δ in $[T]$ and Φ is continuous.
(ii) If $\Theta : G \to [S]$ is continuous, where G is a G_δ subset of $[T]$, then there is a monotone map $\Phi^* : T \to S$ such that $\Theta = \Phi$.

Proof: (i) The most natural way to write $D(\Phi^*)$ as the intersection of an infinite family of sets is to consider the family of G_m, where

$$G_m = \{\alpha | \alpha \in [T] \ : \ \exists n^m \in \mathbb{N} \quad l(\Phi^*(n^m_\alpha)) > m\}.$$

Obviously $D(\Phi^*) = \bigcap_m G_m$.
If α is in G_m, then $l(\Phi^*(n^m_\alpha)) > m$. If $\beta \succ n^m_\alpha$, then β is in G_m, therefore $B(n^m_\alpha) \subseteq G_m$, which says that G_m is open.
To show that Φ is continuous it suffices to show that $\Phi^{-1}(B(w))$ is open in $[T]$, where $w \in S$. Since,

$\alpha \in \Phi^{-1}(B(w)) \Leftrightarrow \Phi(\alpha) \in B(w) \Leftrightarrow \bigcup_n \Phi^*(n_\alpha) \in B(w)$
$\Leftrightarrow \exists_k \ : \ \bigcup_{n=1}^k \Phi^*(n_\alpha) \succeq w \Leftrightarrow \exists_k \ : \ \Phi^*(k_\alpha) \succeq w,$

then $D(\Phi^*) \cap B(k_\alpha) \subseteq \Phi^{-1}(B(w))$, and $D(\Phi^*) \cap B(k_\alpha)$ is open in $D(\Phi^*)$.
(ii) We shall prove here only the proper case where $G = [T]$, since this is the case we need in Chapter 2[23]. Note that $[T]$ is trivially G_δ. In $(*)$ there is a connection between

[22] All the fact on trees which we do not prove here are shown in the Appendix, where $X = \mathbb{N}$.
[23] The proof of the general case can be found in [Kechris 1995] p.8.

u and w, but it is possible that $(*)$ is true for u and $w^{'}$, where $w^{'} \neq w$. It is therefore natural to define for a fixed $u \in T$ the set

$$\Omega(u) = \{w | w \in S \ : \ \gamma \succ u \Rightarrow \Theta(\gamma) \succ w\}.$$

It is trivial to check that

(a) $<> \in \Omega(u)$, therefore $\Omega(u)$ is non-empty.
(b) $\Omega(u)$ is a subtree of S.
(c) $w, w^{'} \in \Omega(u) \Rightarrow w \preceq w^{'} \ \vee \ w^{'} \preceq w$.
(d) $u \preceq u^{'} \Rightarrow \Omega(u) \subseteq \Omega(u^{'})$

We define Φ^* as follows:

$$\Phi^*(u) = \begin{cases} w_0 & \text{, if } \exists w_0 \in \Omega(u) : l(w_0) = l(u) \\ sup\{w | w \in \Omega(u)\} & \text{, otherwise} \end{cases}$$

Φ^* is well defined since, if $\exists w_0 \in \Omega(u) : l(w_0) = l(u)$ i.e., if w's in $\Omega(u)$ are at least as long as u, then there is only one w_0 with above property because of (c). If there is no such w_0 i.e., if w's in $\Omega(u)$ are short with respect to u, then $\Omega(u)$ is finite and $sup\{w|w \in \Omega(u)\}$ is well defined (if $\Omega(u)$ was infinite, then there would be w's in $\Omega(u)$ arbitrarily long, and then, by (b), there would be a w of length $l(u)$). In both cases

$$\Phi^*(u) \in \Omega(u).$$

Monotonicity of Φ^* is a direct consequence of (d) in both cases of its definition.
We need to show that, if α is in $[T]$, then the sequence $\Phi^*(n_\alpha)$ is not stagnant. Suppose that it is i.e., $(\exists n_0 \in \mathbb{N})(\forall n \geq n_0)\Phi^*(n_\alpha) = w_1$, for some $w_1 \in \Omega(u)$. Consider n such that $n > n_0$ and $n > l(w_1)$. For such n, $w_1 = sup\{w|w \in \Omega(n_\alpha)\}$, since n_α is longer than w_1. Consider now node w, such that $w_1 \prec w \prec \beta$. By *continuity* of Θ, there is $m \in \mathbb{N}$ such that $\gamma \succ m_\alpha \Rightarrow \Theta(\gamma) \succ w$ i.e., $w \in \Omega(m_\alpha)$.
If $m < n_0$, then $\Phi^(m_\alpha) = w|m$. But since, by (d), $\Omega(m_\alpha) \subseteq \Omega(n_\alpha)$, $w \in \Omega(n_\alpha)$ and w_1 is not $sup\{w|w \in \Omega(n_\alpha)\}$, which is absurd.
If $m \geq n_0$, then $\Phi^(m_\alpha) = w_1$. Since $w \in \Omega(m_\alpha)$, then by (d), w_1 is not $sup\{w|w \in \Omega(n_\alpha)\}$, for an n, such that $n > n_0$ and $n > l(w_1)$, something which is again absurd.

Finally, we show that $\Theta = \Phi$ i.e., $\Theta(\alpha) = \bigcup_n \Phi^*(n_\alpha) = \Phi(\alpha)$. If there was an α such that $\Theta(\alpha) \neq \Phi(\alpha)$, then there exists n, such that $\Phi^*(n_\alpha) \not\prec \Theta(\alpha)$. But, by definition of $\Phi^*(n_\alpha)$, $\gamma \succ \alpha \Rightarrow \Theta(\gamma) \succ \Phi^*(n_\alpha)$. Since $\alpha \succ n_\alpha$, $\Theta(\alpha) \succ \Phi^*(n_\alpha)$, which contradicts our hypothesis. Therefore, $\Theta = \Phi$.$\diamond$

The above proposition gives the impression that a continuous function $\Theta : [T] \to [S]$ is less abstract object than an arbitrary function $\Theta : [T] \to [S]$, since Θ determines Φ^* which *computes* Θ. But this determination is abstract itself, since the definition of $\Omega(u)$ is far from easy to actually operate. I.e., it is non-trivial to show that a node w of S belongs to $\Omega(u)$, since, in general, it is impossible to check in finite time that every $\gamma \succ u$ has the property $\Theta(\gamma) \succ w$.

A **fan**, or *a finitely branching tree*, is a tree each node of which has a finite number of immediate successor nodes. A *subfan* G of a fan F is just a subtree of F; then G is also a fan. Obviously, Cantor trees are fans.
If T is a tree and A is any subset of $X^{\mathbb{N}}$, we define $< A >$ to be the set of nodes of T which precede some node of A i.e.,

$$< A > = \{u | u \in T \ : \ \exists w \in A \ w \succeq u\}.$$

Then, it is clear that:
(i) $< A > \preceq T$.
(ii) If T is a fan, then $< A >$ is also a fan.
(iii) If $A \subseteq T$, then $A \subseteq < A >$ and $< A >$ is the least subtree of T containing A.

If T is an arbitrary tree such that each branch of T is finite, then T has not in general a branch of maximum length. E.g., the tree $\mathcal{T}$ on $\mathbb{N}$, for which the constant sequences

$$(1), (2,2), (3,3,3), ..., (\underbrace{n, n, ..., n}_{n}), ...$$

are its nodes, is such an infinitely branching tree.
If a tree F though, is a fan with all its branches being finite, then there is always a branch of F of maximum length. This is the content of König's lemma (KL)[24].

We show first that König's lemma has the following equivalent formulations.

Proposition 2.3: If F is a fan on X, then the following are equivalent:

KL_1: If every branch of F is finite, then F has a branch of maximum length.
KL_2: If F has no branch of maximum length, then F has an infinite branch α.
KL_3: If F has a branch of each finite length n, then F has an infinite branch α.
KL_4 (Unendlichkeitslemma): If F has infinite number of nodes (i.e., if F is infinite), then F has an infinite branch α.

Proof: $KL_1 \Rightarrow KL_2$: By contraposition on KL_1.
$KL_2 \Rightarrow KL_3$: If F has a branch of each finite length n, then F has no branch of maximum finite length.
$KL_3 \Rightarrow KL_4$: If F has infinite number of nodes and there is n such that no node of F has level n, then there is no node of F with level $> n$, since F is closed under initial segments. Hence, all the infinite nodes of F have level $< n$. But then, since F is a fan, F is finite, which is a contradiction.
$KL_4 \Rightarrow KL_1$: By contraposition on KL_4, if there is no infinite branch, then F is finite. Therefore, there is a branch of maximum finite length.$\diamond$

We prove now König's lemma in the form KL_4.

Proof of König's lemma: (König's initial proof) We call a node of F *good* iff it has infinite descendants. The root of F is good, since F is infinite. A node of F is called *bad* iff it has a finite number of descendants.
If all the immediate successor nodes of a node u are bad, then u is also bad, since F is a fan.
Hence, by contraposition, a good node has an immediate descendant node which is also good. So, the root $<>$ of F has a good successor node (α_0), which in turn has a good successor node (α_0, α_1), etc. By that way an infinite branch

$$(\alpha_0, \alpha_1, \alpha_2, ..., \alpha_n, ...)$$

of F is formed.$\diamond$

In the end of the above proof we used the, weaker to the Axiom of Choice (AC), Principle of Dependent Choices (PDC):

[24] König's lemma (1926) was used in the proof of a generalization of the Cantor-Schröder-Bernstein theorem. Its story can be found in [Franchella 1997].

For each set A and binary relation $P \subseteq A \times A$ on A,

$$(a \in A) \ \& \ (\forall x \in A)(\exists y \in A)(xPy)$$
$$\Rightarrow (\exists f : \mathbb{N} \to A)(f(0) = a) \ \& \ (\forall n \in \mathbb{N})(f(n)Pf(n+1)).$$

If A is the set of good nodes of F, $a = <>$, and P on A is the relation S of F, by PDC, the whole infinite branch α, and not an as large as we want initial segment of α, is formed.

In that way the infinite branch α exists in the absolute sense of the infinite[25].

Consider the tree T on $\mathbb{N}$ the only nodes of which are the 1-nodes $(1), (2), ..., (n)....$ Obviously, T satisfies the content of KL_1 but it is not a fan, since the root has infinitely many immediate successor nodes. Therefore, the inverse of KL is not true. Also, T does not satisfy the content of KL_4, therefore, KL_4 is equivalent to KL_1, KL_2, KL_3, only if T is a fan.

The argumentation in the proof of König's lemma justifies the following, more general scheme, which we call **König's scheme** (KS):

Let F be an infinite fan and $G(u)$ a predicate on nodes.
(i) $G(<>)$, and
(ii) $[(\forall u \frown k \in F) \ \neg G(u \frown k)] \Rightarrow \neg G(u)$
Then, $(\exists \alpha \in [F])(\forall n \in \mathbb{N}) \ G(n_\alpha)$.

Obviously, if we define for an infinite fan F the predicate

$$G(u) \equiv u \text{ is a good node}$$

then, $KS \Rightarrow KL$, since a node is good or bad, bad being the negation of good.

That F has to be infinite in KS can be seen by considering a tree with a finite number of immediate successors of the root and no other nodes. If we define

$$G(u) \Leftrightarrow \neg[(\exists v \succ u) l(v) = l(u) + 2],$$

[25] A proof can be given through a consequence of PDC, the principle of countable choices (PCC). According to it, if $R \subseteq \mathbb{N} \times A$, a binary relation on $\mathbb{N}$ and set A,

$$(\forall n \in \mathbb{N})(\exists a \in A)(nRa) \Rightarrow (\exists f : \mathbb{N} \to A)(\forall n \in \mathbb{N})(nRf(n)).$$

Knowing that each good node of the fan F has a good successor node and by the existence of an enumeration (a_n) of the nodes of F (the enumeration is possible, since F is a fan), we define the following $R \subset \mathbb{N} \times A$:

$$R(n) = \left\{ \begin{array}{ll} a_0 & \text{,if } a_n \text{ is bad} \\ gs(a_n) & \text{,if } a_n \text{ is good} \end{array} \right\},$$

where $gs(a_n)$ is a good successor node of a_n. By PCC, there is an $f : \mathbb{N} \to A$ such that $nRf(n)$. The infinite branch of F is:

$$\begin{gathered} b_0 = a_0 \\ b_1 = f(a_0) \\ b_2 = f(\delta(b_1)) \\ .... \\ b_n = f(\delta(b_{n-1})) \\ .... \end{gathered}$$

where $\delta(b_n)$ is the index of b_n and the indices of b_n are with respect to the fixed enumeration of the nodes of F.

then G satisfies the hypotheses of KS but the specific tree cannot have an infinite branch. So, KS applies necessarily to infinite fans. Note also, that while the property of good node satisfies

$$(G(u) \ \wedge \ v \prec u) \Rightarrow G(v),$$

this property, which turns the nodes u of F satisfying G into a subtree of F, is not necessary to the proof of König's lemma. Property G of the above counterexample does not, generally, satisfy it.

Although it is not evident, KS is equivalent to an induction scheme, Bar induction, which proves a consequence of KL on trees, the Bar theorem on trees. Bar theorem and Bar induction are classical formulations, in the languages of trees, of Brouwer's fundamental Bar theorem and Kleene's Bar induction scheme, which codifies Brouwer's proof of Bar theorem[26], the same way KS codifies König's proof of KL.
A subset B of a tree T is called a **bar** of T iff each infinite branch of T cuts B i.e.,

$$(\forall \alpha \in [T])(\exists n \in \mathbb{N}) n_\alpha \in B.$$

A *sub-bar* B_0 of a bar B of T is a subset of B which is also a bar of T.

Proposition 2.4 (Bar theorem on fans (BTF)): If B is a bar of a fan F, then B has a finite sub-bar.

Proof: Let B_0 be the set of those nodes of B with no proper initial segment also in B i.e.,

$$B_0 = \{w | w \in B \ : \ (u \prec w) \Rightarrow u \notin B\}.$$

B_0 is called the *thin* bar contained in B. As we have already remarked, $< B_0 >$ is a subfan of F and $B_0 \subseteq < B_0 >$. If $< B_0 >$ has an infinite branch α, then α cuts B_0 at $n_\alpha = w$. Since $(n+1)_\alpha \in < B_0 >$ too, then, by the definition of $< B_0 >$, there is some $w^{'} \in B_0$ such that

$$w \prec (n+1)_\alpha \preceq w^{'},$$

which is absurd, since B_0 is thin.
By contraposition in KL_4, $< B_0 >$ is finite, therefore, B_0 is also finite.$\diamond$

Bar theorem is considered to be a "constructive" version of König's lemma. This constructive character though, is not at all present here. In Paragraph 3 we comment on the non-constructive character of König's lemma. A constructive character is generated in Brouwer's interpretation of the related concepts and in Brouwer's proof of Bar theorem. The Brouwerian induction scheme which codifies his proof of Bar theorem becomes here the **Bar induction on fans (BIF)**.

If F is an infinite fan[27] and B, W are predicates on the nodes of F such that:

(i) $(\forall \alpha \in [F])$ $(\exists n) B(n_\alpha)$, and
(ii) $(\forall u \in F)$ $B(u) \Rightarrow W(u)$, and
(iii) $[(\forall u \frown k \in F)$ $W(u \frown k)] \Rightarrow W(u)$,
Then, $W(<>)$.

Condition (i) expresses the fact that the set of nodes u of F such that $B(u)$ is a bar of F. BIF expresses a kind of backward induction for $W(u)$, going from the validity of W

[26]We study these results in Paragraph 13.
[27]If we replace F by an arbitrary tree we get the bar induction scheme on trees (BIT).

on the nodes of F down to the validity of $W(<>)$.

Proposition 2.5: $BIF \Rightarrow BTF$.

Proof: If B is any bar of F we define the following predicate $W(u)$, $u \in F$:

$$W(u) \equiv (u \in B) \ \vee \ (\exists B_0 \subseteq B : (\forall \alpha \succ u)(\exists n \in \mathbb{N}) \ \ n_\alpha \in B_0).$$

$W(u)$ trivially satisfies condition (ii) of BIF. Also, if $(\forall u \frown k \in F)\ W(u \frown k)$, then there is a finite subset B_0^k corresponding to each $u \frown k$. Since there are finite only nodes $u \frown k$ in F which extend u, then the set

$$B_0^u = \bigcup_{u \frown k \in F} B_0^k$$

is a finite subset of B and $W(u)$ is satisfied. If some nodes $u \frown k$ belong to B then B_0^u is formed by the union of those $\{u \frown k\}$ with all the rest sets B_0^k. Again, if $u \notin B$, B_0^u satisfies the definition of $W(u)$-validity.
The conclusion of BIF says that there is a finite subset B_0 of B such that each sequence which extends the root cuts B_0. Since every F-sequence extends the root, $W(<>)$ expresses the fact that the subset B_0 which corresponds to the root is a finite subfan of B. If $W(<>)$ is interpreted as $<> \in B$, then $\{<>\}$ is the finite subfan of B in question.$\diamond$

Proposition 2.6: $KS \Leftrightarrow BIF$.

Proof: ($KS \Rightarrow BIF$) Suppose (i)-(iii) of BIF and also the negation of BIF's conclusion i.e., $\neg W(<>)$. If we define

$$G(u) \equiv \neg W(u),$$

where $u \in F$, then the (i) and (ii) of KS are trivially satisfied, since $\neg\neg W(<>) \Leftrightarrow W(<>)$. Therefore, $\exists \alpha \in [F]\ G(n_\alpha), \forall n \in \mathbb{N}$, or equivalently, $\neg W(n_\alpha), \forall n \in \mathbb{N}$. But, α necessarily cuts the bar B, i.e., $\exists m \in \mathbb{N}$ such that $m_\alpha \in B$, therefore, by (ii) of BIF, $W(m_\alpha)$, which is a contradiction. Hence, given KS, the hypothesis $\neg W(<>)$ is contradictory to the hypotheses of BIF, therefore $\neg\neg W(<>)$ holds, which classically gives $W(<>)$.

($BIF \Rightarrow KS$) Assume hypotheses (i) and (ii) of KS and the (classical) negation of its conclusion i.e., $(\forall \alpha \in [F])(\exists n \in \mathbb{N})\neg G(n_\alpha)$. We define

$$W(u) = B(u) \equiv \neg G(u),$$

where $u \in F$. Then, hypotheses (i)-(iii) of BIF are satisfied. Hence, $W(<>) \Leftrightarrow B(<>) \Leftrightarrow \neg G(<>)$, which contradicts hypothesis (i) of KS. Therefore, $(\exists \alpha \in [F])(\forall n \in \mathbb{N})G(n_\alpha)$.$\diamond$

So, though it was not at all clear at the beginning, KS is *classically* a kind of induction, namely the backward kind of induction of BIF. This is not intuitionistically true, since

$$\neg\neg P \rightarrow P,$$

which is used in both directions of the above proof, is only classically accepted. In later paragraphs we see that this is not the only problem from an intuitionistic point of view. The above results are described with the following diagram:

$$
\begin{array}{ccc}
KL & \Rightarrow & BTF \\
\Uparrow & & \Uparrow \\
KS & \Leftrightarrow & BIF
\end{array}
$$

König's lemma is connected to compactness. E.g, the compactness theorem of Propositional Calculus can be proved through KL (see e.g., [Smullyan 1968], pp.30-34). KL is also used in proving that if T is a pruned tree, $[T]$ is compact iff T is a fan[28]. Also, a proof of Ramsey theorem can be given through KL (see [Simpson 1999] p.123).

3. Brouwer's Fundamental Principle. In the proof of König's lemma we used a standard criterion of existence, in order to prove the existence of the infinite branch $(\alpha_0, \alpha_1, \alpha_2, ..., \alpha_n, ...)$, which we call **Principle of Logical Existence** (PLE). According to it,

A mathematical object, satisfying some property P, exists if the hypothesis of its non-existence leads to an absurdity.

In symbols,

$$\exists_{lo} x P(x) \equiv \neg \exists x P(x) \Rightarrow \bot,$$

where $\bot$ denotes absurdity. Using PLE we proved the existence of a good successor node $\alpha(0)$ of the root, without being able though to indicate which one of the successor nodes of the root really is good. The same method was used in each subsequent step of the formation of the infinite branch. Hence, within PLE it is possible to prove the existence of mathematical objects without being able to construct them or indicate a secure method finding them. PLE is in direct contrast to the **Principle of Constructive Existence** (PCE). According to it,

A mathematical object exists iff it has been constructed with an accepted constructed method[29].

In symbols,

$$\exists_{co} x P(x) \equiv K(x) P(x),$$

where $K(x)$ denotes an accepted construction of the object x, within a constructive theory T_K. Since $K(x)$ depends on the T_K, it would be more accurate to talk about constructive existence within a certain constructive theory T_K. A major part of our study is to clarify $K(x)$ within Brouwer's Intuitionistic Analysis (BIA), a modern constructive theory of the mathematical continuum.

Of course, a non-constructive theory T, which uses PLE, may also use constructive methods allowing constructive proofs of existence. If $K(x)$ is within any kind of theory, then

$$\exists_{co} x P(x) \Rightarrow \exists_{lo} x P(x),$$

but not conversely. The logical existence of the infinite branch of König's lemma in no way implies a method of its construction. On the contrary, it seems impossible to find a way to construct it. If we follow the evolution of the fan from the root we check all finite

[28]We actually give this proof in the characterization of compact sets of $\mathcal{N}$ (See the Appendix, Proposition A.6).

[29]It is non-trivial to say which constructive method is the right one. A classical example from antiquity is the use of neusis in geometric constructions. Neusis, as a constructive method, is stronger than line and circle but its use was doubted from the beginning (see [Bos 2001]).

nodes of each one level. Some of these nodes may end, and some of them necessarily continue to grow, since otherwise the root would be a bad point. The logically existed infinite branch is an infinite extension of one of these growing nodes, but we cannot tell of which one. In Paragraph 13 we show that König's lemma is unacceptable within BIA, although we cannot show that it is false within BIA[30]. In general, we cannot show the negation of the inverse of $\exists_{co} x P(x) \Rightarrow \exists_{lo} x P(x)$ i.e.,

$$\neg\neg[\exists_{lo} x P(x) \Rightarrow \exists_{co} x P(x)].$$

PLE presupposes an abstract world W of mathematical objects, which is consistent and the Principle of the Excluded Middle is true in it. Since

$$\exists x P(x) \ \vee \ \neg\exists x P(x),$$

the proof of $\neg\neg\exists x P(x)$, i.e., of $\neg\exists x P(x) \Rightarrow \bot$, entails that $\exists x P(x)$ in W. I.e., in W

$$PEM \Rightarrow PLE.$$

Brouwer is famous for his disbelief to PEM and his non-classical interpretation of logical connectives. If u is a node of a fan F, then classically

$$G(u) \ \vee \ \neg G(u)$$

is true, but intuitionistically we must be able to say which one of the disjuncts is actually true, something which is, in general, impossible[31].

To a constructivist like Brouwer logical existence conveys all the epistemological problems of world W. Since it cannot be adequately explained how the human mind is connected to W and PEM in W, PLE is not sufficient to guarantee the actual existence of a mathematical object.

It is important to stress here that $K(x)$ is far more complicated enterprize from just defining x. As we have already seen in the proof of Proposition 2.2, the definition of $\Omega(u)$ is not constructive, since there is no general method constructing even a single element of $\Omega(u)$. A definition of a mathematical concept is at first a linguistic expression without a genuine mathematical meaning, unless an appropriate construction accompanies it.

Logical existence within a non-constructive theory T though, is not without value to a constructive theory T_K with analogous objects to T. Logical existence of an object x may serve as a guide to find a constructive proof of it.

In antiquity all existence proof were, roughly, constructive and until the end of 19th century constructive spirit was still alive. It is no strange that Bolyai tried to give a geometric line and circle construction of the limiting parallel in order to justify the new concept[32]. It was because of the alive constructive spirit of the 19th century that Gordan said "Das ist nicht Mathematik, das ist Theologie", regarding Hilbert's non-constructive proof of basis theorem[33]. This remark forced Hilbert to find a constructive proof of it. Bishop though, in [Bishop 1968] pp.55-56, remarks:

[30] In recursive analysis, where all sets must be recursively defined it can be shown that KL is false (see e.g., [Beeson 1985] p.68).

[31] See also on that [van Atten 2004] p.63 and [Dummett 2000] pp.49-51.

[32] The problematic character of Bolyai's construction is discussed in [Hartshorne 2000] pp.396-398 and in [Petrakis 2008].

[33] If R is a Noetherian ring, then $R[X]$ also is (see e.g., [Kendig 1977] pp.118-121) for a classical proof.

[Recently I was asked whether the Hilbert basis theorem ... is constructively valid. The answer is easily seen to be 'yes'. Unfortunately, not even the ring of integers is Noetherian from the constructive point of view (and therefore the Hilbert basis theorem is vacuous). For a counterexample in the style of Brouwer, let $\{n_k\}$ be a sequence of integers, for which we are in doubt as to whether they are all equal to 0. The ideal generated by the integers n_k has no finite basis in the constructive sense. The problem is to find a constructively usable reformulation of the definition of a Noetherian ring, which would include the integers and give constructive substance to the Hilbert basis theorem.]

Even if Hilbert was not the first to give a non-constructive proof, he was a major proponent of the non-constructive spirit, especially in his early period. Although Kronecker, who in general influenced Hilbert[34], believed that existential propositions are meaningless if they do not explicitly specify the object the existence of which they ascertain, Hilbert saw in the negation of PLE a major shrink of mathematics. Providing a constructive proof of his previously non-constructively proven basis theorem, he revealed the importance of PLE, since the object that had to be constructed was already proven to "exist". Even Gordan admitted that "theology" had its merits. Of course, as we can already suspect from König's lemma case, this cannot be done with every non-constructively proven theorem.
An advantage of the use of PLE was the at least quantitative development of mathematics, permitting the introduction and use of objects that was hard or impossible to construct. The marginalization of constructivism though, begun quite earlier. E.g., the algebraic determination of a curve through its equation, which replaced the sometimes hard to find geometric construction of it begun in the 17th century (see [Bos 1993]). Though this change of point of view gave a new impetus to the study of curves, it had a serious philosophical cost. The curve by being equated to its equation stopped being a truly continuous object demanding an appropriate construction. Gradually, the discrete approach to continuous objects (we may know as many as possible discrete points of a curve and within Infinitesimal Calculus how it, roughly, looks) replaced the geometric construction of a curve as certain foundation of our knowledge of it. Finally, the answer to the question of mathematical existence became of non-constructive character. In that way problems could be also solved easier.
Though Brouwer was completely against mathematics with PLE, it is worth remarking that by

$$\neg\exists_{lo} x P(x) \Rightarrow \neg\exists_{co} x P(x),$$

the proof of logical non-existence implies constructive non-existence. In that way constructively acceptable results of logical non-existence can be incorporated to a T_K.
There are two major, classical questions on the philosophy of mathematics.

The ontological question on mathematical objects(OQM): Which is the nature of mathematical objects?

The epistemological question on mathematical objects (EQM): How do we know mathematical objects?

The tradition of constructive mathematical existence is connected to a fundamental

[34] See Hilbert's obituary by Weyl in [Weyl 1944].

principle, which, though not explicitly expressed, was on the ground of mathematical practice for many centuries. It is the fundamental principle of a geometric constructive framework $\mathfrak{G}$, the basic principles of which we present in [Petrakis 2010].

Fundamental Principle of $\mathfrak{G}$ ($FP_{\mathfrak{G}}$): Mathematical objects, except some initial mental intuitions, are constructions of the human mind, based on these initial intuitions.

For centuries mathematical objects were to mathematicians (and philosophers e.g., like Kant) creations of the human mind based on certain mind intuitions. The only way then that a non-fundamental object exists is to be constructed appropriately by the fundamental intuitions. $FP_{\mathfrak{G}}$ answers simultaneously both major questions. Mathematical objects are mental intuitions, fundamental or not, and we know them because there are part of our mind (fundamental intuitions) or because we construct them (non-fundamental objects) by the fundamental ones. Therefore constructive existence is a result of the nature of mathematical objects i.e.,

$$FP_{\mathfrak{G}} \Rightarrow \exists_{co}(x)P(x).$$

The "discovery" of non-Euclidean geometries, which undermined our faith to the fundamental intuition of space and the parallel arithmetization of analysis turned mathematicians away from $FP_{\mathfrak{G}}$. Gradually, the OQM was answered through W within which EQM is completely neglected. The only problem left was the consistency of W (early formalism of Hilbert).
In such a foundational atmosphere Brouwer, although influenced by it, built a bridge with traditional constructivism.

Brouwer's early Fundamental Principle (BFP_1) is exactly the same to $FP_{\mathfrak{G}}$. In his late period he used a variation of it, BFP_2, according to which, mathematical objects, except some initial mental intuitions, are constructions of the human mind *of the ideal mathematician*, based on these initial intuitions.

The concept of the mind of the ideal mathematician, a mathematical subject of great mathematical memory and patience, is an idea of Brouwer's mature period. In his early period his fundamental principle is an independent rediscovery of $FP_{\mathfrak{G}}$. Already in his dissertation[35] he claims that:

> [... to exist in mathematics means to have been constructed by intuition.]

Although Brouwer deviated from $\mathfrak{G}$ in the nature of the fundamental intuitions, his idea that mathematics is the constructive product of some fundamental intuitions is crucial to the development of his reconstruction of mathematics.
The whole third chapter of his dissertation is dedicated to the unacceptability of all mathematical objects built independently from intuition ([Brouwer 1907], p.52). With his critique on the axiomatic foundation of mathematics, on Cantor's theory of transfinite numbers, on Peano-Russell's logicism and mainly on Hilbert's early formalism, he tried to explain why the only possible real foundation of mathematics was within his fundamental principle. Even the set-theoretical paradoxes of that period are treated by Brouwer as symptoms of the deviation of set theory from his fundamental principle and its consequences.

[35] [Brouwer 1907] p.96.

Even if W was proven consistent, PLE had to be denied ([Brouwer 1907] p.79), and PCE had to be accepted as a direct consequence of BFP_1. I.e.,

$$BFP_1 \Rightarrow \exists_{co}(x)P(x).$$

Intuitionistically, a non-fundamental mathematical object is a legitimate mental construction, therefore, in order to show its existence, we must show its construction. In that way, both in $\mathfrak{G}$ and BIA, *the ontology and epistemology of mathematical objects are identified.* We show the existence of a non-fundamental object by showing how we know it.

Brouwer was not the only one with such foundational views. Weyl in "Das Kontinuum" holds similar to Brouwer views on the relation between language and genuine mathematics and on the foundational use of the axiomatic method. Hilbert's early axiomatic method was also criticized by Frege and Poincaré. It was only Brouwer though, who developed the mathematical consequences of his fundamental principle and insisted on it. In our days both $FP_{\mathfrak{G}}$ or BFP_1, and BFP_2 are unpopular.

4. The Second Act of Intuitionism: choice sequences, creating subject and species. According to Brouwer (who echoes Kant), the natural numbers are certain mental constructions founded on the primordial intuition of time. This philosophical stand of Brouwer is called the First Act of Intuitionism (FAI)[36]. Brouwer founded the natural numbers on the intuition of time two-ity, a pair of time moments, and defined rational numbers by naturals in a classical way. All expected properties hold for intuitionistic rationals.

While FAI determined the discrete intuition of time as a foundational basis (see [Brouwer 1907]), the **Second Act of Intuitionism (SAI)** (formulated for the first time in 1918) determined the ways by which new objects are constructed by already existed or constructed ones, in order to built BIA. According to Brouwer's own words ([Brouwer 1952] p.142),

> [The second act of intuitionism recognizes the possibility of generating new mathematical entities:
>
> firstly in the form of **infinitely proceeding sequences** $p_1, p_2, ...$, whose terms are *chosen more or less freely from mathematical entities previously acquired*; in such a way that the freedom of choice existing perhaps for the first element p_1 may be subjected to a lasting restriction at some following p_ν, and again to sharper lasting restrictions or even abolition at further subsequent p_ν's, while all these restricting interventions, as well the choices of p_ν's themselves, at any stage may be made to depend on future mathematical experiences of the creating subject ;
>
> secondly in the form of mathematical **species**, i.e. *properties supposable for mathematical entities previously acquired*, and satisfying the condition that, if they hold for a certain mathematical entity, they also hold for all mathematical entities which have been defined to be equal to it, relations of equality have to be symmetric, reflexive and transitive; mathematical

[36]For the mental construction corresponding to a natural number and the derivation of Peano "axioms" by this construction see [Petrakis 2007].

entities previously acquired for which the property holds are called *elements* of the species.]

Within SAI, Brouwer explains how new objects are constructed from old ones, in a way similar to that of Euclid's postulates, which determine all constructions of geometric objects.
SAI is highly non-trivial, since through it Brouwer transcends the reduced continuum $\mathcal{R}$, in which all points are defined through a law-like Cauchy sequence of rationals, therefore $\mathcal{R}$ is countable, and creates full intuitionistic continuum, which is not countable.
Brouwer believed that classical continuum is not a legitimate object, but only a linguistic expression, in contrast to the intuitionistic continuum.
We intend in the following paragraphs to clarify these facts.
The first constructive mechanism in BIA is "sequencation", i.e., an infinitely proceeding sequence of already existed objects $p_1, p_2, ...$, is a new object $(p_1, p_2, ...)$. In symbols,

$$p_1, p_2, ... \hookrightarrow (p_1, p_2, ...)$$

What is new in this construction principle is that an incomplete, not necessarily predetermined object like $p_1, p_2, ...$, is accepted as a "genuine" object. Since $p_1, p_2, ...$, is not predetermined it is called a **choice sequence**. The term "choice" sequence will be understood through the concept of spread, the Brouwerian concept which houses choice sequences.
The incompleteness in our knowledge of a choice sequence α, since α is ever growing without knowing exactly how it grows, makes α difficult to accept. But Brouwer insisted on its use and the reason for that will be clear only after showing the merits of the spread concept. Choice sequences are not accepted neither in classical mathematics nor in other constructive theories[37].
Borel introduced choice sequences in a lecture of 1908 (see [Borel 1909]) attended by Brouwer. There he discusses the possibility the uncountability of choice sequences to count for continuum, but denies choice sequences as legitimate objects. In 1912 Borel writes[38]:

> [People will also agree on the following point: it is possible to define a decimal number of bounded length by asking thousand people to write down, arbitrarily, some digit; thus one obtains a well-defined number, if all the persons are arranged in a row, and each one writes in turn a new digit at the end of the sequence of the digits already written by the people in the row preceding him. But observe where the disagreement sets in: it is possible to define a decimal number of unbounded length by a similar process?...
>
> On my part, I regard it as possible to ask questions of probability concerning decimal numbers obtained in this way, by choosing digits, either entirely arbitrarily, or imposing some restrictions which leave some arbitrariness, but I regard it as impossible to talk about a single individual such number,

[37]In Russian constructivism of Markov sequences are only defined recursively, while in Bishop mathematics choice sequences are dismissed, making mathematics "so bizarre it becomes unpalatable to mathematicians" (see [Bishop 1967] p.6).

[38]See [Borel 1912].

since if one denotes such a number by a, different mathematicians, in talking about a, will never be sure to be talking about the same number.]

Borel didn't accept choice sequences because equality of choice sequences is undecidable. It is impossible, in general, to find a method that decides in finite time if two choice sequences are equal. Weyl, who followed in early twenties Brouwer's views couldn't accept choice sequences as genuine objects for the same reason. Brouwer, who followed at the beginning Borel's view, changed his mind and by 1914 onwards he gave to choice sequences an object status.
Until then, the only concept of sequence was that of a law-like sequence, given by a certain rule f which determines all elements $f(1), f(2), ..., f(n), ...$, of the sequence, or that of an abstract sequence, which belongs to the external to mind mathematical world W and it is, therefore, independent from our knowledge of it.
Although Brouwer didn't have a single concept of choice sequence in his mind throughout his life[39], he worked mainly with choice sequences within a spread. As we explain in Paragraph 5, a spread is a kind of a non-deterministic rule, with the help of which the **creating subject** (CS) selects the terms of choice sequences. CS is Brouwer's mature addition to his early fundamental principle BFP_1. I.e.,

Brouwer's mature Fundamental Principle (BFP_2): Mathematical objects, except some initial mental intuitions, are constructions of the creating subject, based on these initial intuitions of his.

Hence, mathematics is the mathematical activity of an idealized human mind, having:
(i) **Perfect memory**, so that he remembers all of his past actions,
(ii) **Great patience**, so that he is engaged in ω-procedures, something which a normal person never considers.
(iii) **Will** to interfere in a mathematical procedure. Since mathematics is CS's activity by definition, CS choses one object among others and may decide to stop or wait until some condition is satisfied.
(iv) **Knowledge** or **ignorance** on certain mathematical questions, which CS may incorporate to his mathematical activity.
(v) **Grasp of all fundamental intuitions**, on which mathematical activity is based.
(vi) **No special features** i.e., what a CS does can be done by any CS.

Idealizations (i) and (ii) are of quantitative character only, while properties (iii) and (iv) are related to CS's "situation". Will, knowledge or ignorance, are human properties that are found for the first time in a mathematical theory. Their introduction seems at first peculiar, but if we take BFP_2 seriously, then the use of properties (iii) and (iv) makes sense.
CS is present, directly or not, in fundamental intuitionistic notions. Natural numbers, spread choice sequences and intuitionistic functions reflect constantly the presence of CS.
The main characteristic of spread choice sequences is that they are incomplete, ever growing sequences, therefore their equality is undecidable. A choice sequence is a major example of an on-going mathematical object, which is formed by CS in time (in

[39] For the different kinds of choice sequences in Brouwer's work see [Troelstra 1981]. There is also a letter of Brouwer to Heyting mentioning *lawless sequences*, which are completely independent from any kind of law. Lawless sequences were later introduced by Kreisel for metamathematical purposes.

compatibility to FAI, where discrete time intuition is the fundamental intuition). Creation of a choice sequence in time means creation in CS's subjective (personal) time. Since an ω-procedure is the model of an on-going procedure in time, and since these procedures do not reflect characteristics of a special CS, it is safe to say that construction in time of a spread choice sequence is objective i.e., independent from any special CS. This situation is repeated throughout BIA, so that property (vi) of a CS is satisfied. Since the concept of choice sequence that we study here depends on the notion of spread, we postpone its further analysis until the introduction of spreads in next paragraph.

The second mechanism of generation of new objects from old is that of species.

Standard description of species: A species E of already constructed objects is a property defined on them.

Since E is a property on already existed objects impredicative definitions are avoided[40]. The **order of a species** is defined inductively as follows:

(i) Mathematical objects, like natural numbers, spread choice-sequences, are species of order 0.

(ii) If the already constructed objects on which E is applied are of order n, then E is a mathematical object of order $n+1$.

In that way a hierarchy analogous to the hierarchy of sets in type theory is formed[41]. A species of already constructed objects is a new object which is considered legitimate from the intuitionistic point of view. The **central question on species** is:

Why defining a property E on already constructed objects is enough to accept E constructively?

Although we have not defined yet any species, it is interesting to see what has been said on the central question of species.

Although species belong to Brouwer's mature period, we find a constructive approach on the notion of mathematical property already in [Brouwer 1907] p.52:

> [Often is quite simple to construct inside such a structure, independently of how it originated, new structures, as the elements of which we take elements of the original structure or systems of these, arranged in a new way, but bearing in mind their original arrangement. The so called 'properties' of a system express the possibility of constructing such new systems having a certain connection with the given system.]

Quotes as the above made van Stigt (in [van Stigt 1990] p.337) and van Atten (in [van Atten 2004] p.6) to answer the central question on species through the intuition of two-ity. According to van Atten, the already constructed objects α on which E is referred to and the already constructed objects α which actually satisfy E form a pair, the

[40] A circular or impredicative definition of an object a is one in which a totality A, such that $a \in A$, is used in it. E.g., if the set of naturals is defined as the intersection of all inductive subsets of reals, while naturals belong to the totality of all inductive subsets. Within W circular definitions are accepted, but that is not the case outside W. Poincaré was critical on a crucial circular definition in Zermelo's proof of well-orderability of any set and Russell created type theory in order to avoid circular definitions ([Russell, Whitehead 1910]). Weyl developed predicative mathematics in "Das Kontinuum" (see [Weyl 1918]) and [Feferman 1997b]).

[41] In mid-twenties Brouwer elaborated a more detailed hierarchy of species, which abandoned after the war (see [van Stigt 1990] pp.340-345).

components of which are connected through E. E "holds together" these two distinct systems, a reflection of the initial intuition of two-ity. The CS separates from objects α those which satisfy E and this is a new construction, a new object and at the same time connected to the initial system of objects α. In that way the construction of species is an expression of the unfolding of the initial intuition of two-ity.
Brouwer himself believes that too[42], although he didn't answer explicitly the central question on species. Generally, he believed that SAI is compatible to FAI, since choice sequence and species are reflections of time two-ity, although this reflection in the case of species is quite mysterious. Brouwer himself claimed[43] that consideration of the isolated structure (those α which satisfy E) and the hypothesis of it being part of other constructed entities (those already constructed α on which E is applicable) is a distinct constructive device, a new mathematical entity.
This more or less common "explanation" is not at all persuasive, since in that way all classical properties, only defined on pre-existed objects, are also acceptable and there is no real boundary between classical and intuitionistic properties. As we show in [Petrakis 2010], the analysis of the notion of species is the most crucial in a reconstruction of BIA and the question of the genesis of species needs to be revisited.

Classically, properties define sets. According to Frege's Comprehension Principle, if P is a property, such that $P(x)$ is true or not (without being necessarily decidable), then there is a set X, such that

$$X = \{x|P(x)\}$$

i.e.,

$$x \in X \Leftrightarrow P(x)$$

By extensionality axiom

$$A = B \Leftrightarrow (\forall x)[x \in A \Leftrightarrow x \in B],$$

X is unique and it is called the extension of P, denoted as (P)[44]. Through the famous Russell's property $P(x) \equiv x$ is a set and $x \notin x$, we get for the extension (P) of it and Russell's paradox

$$(P) \in (P) \Leftrightarrow (P) \notin (P),$$

showing the inadequacy of Frege's principle, which turned into Zermelo's separation axiom[45]. According to it, if A is a set and P a property on elements of A, there exists the set X, where

$$X = \{x|x \in A \wedge P(x)\}$$

and obviously

$$x \in X \Leftrightarrow [x \in A \wedge P(x)].$$

[42]See [van Stigt 1990] p.337.

[43]In [Brouwer 1947] p.339 and in [Brouwer 1954] p.2.

[44]Frege studied only sets which are extensions of properties, something which is not the case in axiomatic set theory, where there are sets, like the infinite set determined by the infinity axiom, which are not extensions of properties.

[45]Zermelo, who had found Russell's paradox even earlier than Russell, was that period in Göttingen and was aware of Hilbert's ideas on the value of the axiomatic method.

By extensionality X is unique. Although separation axiom is a restriction of Frege's comprehension principle, through which Russell's paradox is avoided[46], it is not explained why this, and not Frege's comprehension principle, captures the meaning of the concept of set.
Brouwer, even from the beginning, treats all set-theoretic paradoxes as consequences of the linguistic approach to the concept of set[47] and denies the separation axiom. In 1919 he writes[48]:

> [The axiom of Comprehension-on the basis of which all things which have a certain property unite into a set (even in the later, modified version given by Zermelo)-is inadmissible and useless; a legitimate basis of mathematics can only be found in a *constructive definition* of set.]

So, a direct response of Brouwer to the central question of species is that there is some kind of construction associated to the definition of a property E on some already constructed objects. For Brouwer the extensions (P) of properties P are linguistic only objects of the external to us world W. Therefore, they are not accompanied, generally, by some mental procedure which guarantees their understanding.
In order that SAI is compatible to FAI, there must be something more in the standard description of species. In [Brouwer 1925] we find an additional element in Brouwer's description of species which separates species from classical properties[49].

Brouwer's normative description of species: A species E of already constructed objects is a property defined on them, which is *conceptually completed*.

van Stigt also remarks[50]:

> [In the Brouwerian universe of mathematics (property) can only be a construction, and this is the interpretation given in [Brouwer 1907, 1908, 1923], where property is a 'construction' or a 'system'.]

An intuitionistic property is actually a pair $(E, K(E))$, where E is the formulated property and $K(E)$ is a construction which accompanies the formulation of E. $K(E)$ is the conceptual completion of property E, the element of difference between classical and intuitionistic property, the additional element to the standard description of species.
Unfortunately, Brouwer's references to $K(E)$ are scarce and, although it is logically necessary, it is not found in the related literarure[51].

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[46]Russell's paradox is avoided as follows: If P is the Russell property, then

$$(P) = \{x | x \in A \ \wedge \ x \notin x\},$$

therefore, since

$$(P) \in (P) \Leftrightarrow (P) \in A \ \wedge \ (P) \notin (P),$$

we simply infer that $(P) \notin A$. We also conclude that for each set A there is a set, (P), which is not in A, hence, there is no such thing as the set of all sets.

[47]See [Brouwer 1907] p.89.

[48]See [van Stigt 1990] p. 336.

[49]It is through this additional element that Heyting's or Weyl's criticism on the concept of species can be confronted.

[50]In [van Stigt 1990] p.336.

[51]Exceptions are some references of van Stigt in [van Stigt 1990] and his stress of Brouwer's constructive understanding of a property in an introductory text of his in [Mancosu 1998] pp.13-14.

In favor of this constructive interpretation of species we add the fact that Brouwer defines species of species (see in the next paragraph the definition of reals). Even by the standard interpretation, the initial species must be *already constructed*, therefore a construction of those species is presupposed.
While the standard interpretation of species is connected to a classical linguistic approach on properties, Brouwer's normative interpretation is in harmony, at least programmatically, with the rest of intuitionism. As in traditional constructivism, in which concepts are not only defined but also constructed, in BIA construction of concepts is reinvented.
Two species differently defined A, B are called *equal*, $A \approx B$, iff $x \in A \Leftrightarrow x \in B$ i.e., iff they are extensionally only equal.
In the rest of our thesis we present all major examples of Brouwerian species. Through them the character of $K(E)$ will be explored. The question whether species fall under Brouwer's normative description will not be addressed here[52].

5. Spreads and fans. The concept of spread is Brouwer's invention to represent the mathematical continuum, a fundamental intuition in early intuitionism. In Heyting's words[53]:

> [From 1918 on Brouwer no longer mentions the continuum as a primitive notion. He can do without it because the spread ... represents it completely, as far as its mathematical properties go.]

A **spread** is a determined through two laws:

(A) the **spread law** Λ, which decides if a finite sequence of natural numbers is accepted or not. Λ distinguishes between accepted and unaccepted finite sequences as follows:
(i) It decides which 1-sequences (of length 1) are accepted.
(ii) If $(\alpha_1, \alpha_2, ..., \alpha_k, \alpha_{k+1})$ is accepted, then $(\alpha_1, \alpha_2, ..., \alpha_k)$ is also accepted.
(iii) If $(\alpha_1, \alpha_2, ..., \alpha_k)$ is accepted, it decides if some sequence $(\alpha_1, \alpha_2, ..., \alpha_k, m)$ is accepted or not.
(iv) If $(\alpha_1, \alpha_2, ..., \alpha_k)$ is accepted, then there is a natural number m, such that the successor sequence $(\alpha_1, \alpha_2, ..., \alpha_k, m)$ is accepted.

Thus, Λ_M determines a tree with its branches corresponding to the admissible by Λ_M finite sequences or *nodes* of M[54]. Actually, properties (i)-(iii) determine an *intuitionistic tree.* By (iv), all paths of the tree are potentially infinite and they are called *(naked) choice sequences of the spread* M. A spread M can be seen as an *intuitionistic pruned tree.*

(B) the **complementary spread law** Γ, which corresponds to any Λ_M-accepted sequence an *already* constructed mathematical object. So, if $(\alpha_1, \alpha_2, ..., \alpha_k, ...)$ is an M-(choice) sequence, then by the following correspondences of Γ_M

$$\begin{gathered}(\alpha_1) \mapsto \beta_1 \\ (\alpha_1, \alpha_2) \mapsto \beta_2 \\ \cdots\cdots\cdots\end{gathered}$$

[52] See [Petrakis 2010] for details.
[53] In [Heyting 1974] p.84.
[54] The above definition does not specify the nature of Λ_M, only its function. This is not a problem, since BIA uses certain spreads and it is independent from a general theory of spreads.

$$\cdots\cdots\cdots\cdots$$
$$(\alpha_1, \alpha_2, ..., \alpha_k) \mapsto \beta_k$$
$$\cdots\cdots\cdots\cdots$$

an M-sequence of mathematical objects, not necessarily naturals, is constructed. Each M-sequence $(\beta_1, \beta_2, ..., \beta_k, ...)$ is an *infinitely proceeding sequence* (i.p.s), of which we know at any moment only a finite initial segment i.e., an M-sequence is an *on-going mathematical object*, which is also referred to as choice sequence of M. The empty sequence is the *root* <> of the tree M. A spread M without a complementary spread law is called *naked*.

Hence, a spread choice sequence is completely different from a classical sequence, which is a complete object under the umbrella of classically accepted absolute infinity[55]. Within the use of potential infinity only, a sequence given by some law $f(n)$ is a completely given object, since we can find any term of it, independently from the others. I.e., f is not constructed in time. On the contrary, a recursively given law for a sequence can be interpreted as an object constructed in time.
A spread choice sequence is by definition constructed in time[56]. Hence, a classical sequence within absolute infinite framework is completely different object than spread choice sequence and this difference reflects all major fundamental differences between BIA and classical analysis.
A major example of a species is the species $[M]$ of M-choice sequences (naked or not), which corresponds to the classical (set) body of a tree M. Due to SAI an M-choice sequence α is a legitimate mathematical object within BIA. The generation of the species $[M]$ is similar to the species of natural numbers ω. Λ_M, like ω, embodies a common mechanism of construction of certain mathematical objects, rather than a property defined on pre-existed objects, since M-choice sequences are not already constructed but under on-going construction. Both $[M]$ and ω can be considered as **fundamental species** which correspond to a common mode of formation of mathematical objects. So, $[M]$ is not a set but a mechanism of construction of sequences and its conceptual completeness derives from the conceptual completeness of Λ_M. We say that a (naked) M-sequence belongs to $[M]$ iff each initial segment of α is Λ_M-accepted, i.e.,

$$\alpha \in [M] \Leftrightarrow \forall n \ , \ n_\alpha \text{ is } \Lambda_M\text{-accepted},$$

but we actually mean *that α falls under the construction mechanism of* Λ_M.
Of course, the expression "$\forall n$, n_α is Λ_M-accepted" is understood within the potential infinity framework. If α is an infinitely proceeding sequence generated, in general, independently from the spread M (i.e., in our study generated by some other spread N), then the question $\alpha \in [M]$ is not decidable, since it is needed infinite time to check if all initial segments of α are M-accepted.
Two infinitely proceeding sequences $(\alpha_1, \alpha_2, ...,)$ and $(\beta_1, \beta_2, ...,)$ are called *equal* iff $\alpha_n = \beta_n$ for each n, and *positively distinct* iff a natural number s can be indicated such that $\alpha_s \neq \beta_s$.
The expression $\alpha \notin M$, means intuitionistically that

$$\alpha \in M \to \perp.$$

[55]Classically, a sequence $f : \mathbb{N} \to X$ of elements of X is the absolutely infinite set of pairs $(n, f(n))$.

[56]As we have already said in Paragraph 5, there are post-Brouwer concepts of choice sequence independently from a spread (see [Troelstra 1977] and [Troelstra, van Dalen 1988b] Ch.12), which can not be generated though, by a common mechanism like spread choice sequences.

Although the question if a node u is Λ_M-accepted or not is decidable, there is no positive description of the fact $\alpha \notin M$. I.e., intuitionistically

$$\alpha \notin M \Rightarrow \exists n_\alpha \text{ such that } n_\alpha \text{ is not } \Lambda_M\text{-accepted}$$

is not generally true. To accept the existence of such an n is non-trivial and it is related to the non-acceptance of Markov's principle from the intuitionistic point of view.

Examples of spreads:

(I) If Λ_M accepts no natural number as an 1-sequence, then there is no M-node nor M-sequence and M is called the *empty spread*.
(II) If Λ_M accepts a fixed natural number n at each step, then $[M] = \overline{n}$, and M is the spread of the stagnant sequence $\overline{n}$.
(III) If Λ_M admits any finite sequence of naturals, then M is the *universal spread*, the body of which is denoted by ω^ω and corresponds to the classical Baire space $\mathcal{N}$ of all sequences of naturals. But intuitionistic ω^ω is not a set, only a species generated by a mechanism of construction of i.p.s.

The notion of spread is one of Brouwer's most important conceptual innovations, since it *holds together* all the M-sequences, without *containing* them as a set. The spread concept derives from Brouwer's need to avoid the concept of absolutely infinite set.

(IV) The most important spread is the *spread of real numbers* $\Re_{Br}$. If we define the rational numbers in the classical way, and fix an enumeration $q_1, q_2, ..., q_n, ...$ of them, then $\Lambda_{\Re_{Br}}$:
(i) accepts any natural number as a successor of the root $<>$.
(ii) accepts $(\alpha_1, \alpha_2, ..., \alpha_n)$, if it accepts $(\alpha_1, \alpha_2, ..., \alpha_n, \alpha_{n+1})$.
(iii) accepts $(\alpha_1, \alpha_2, ..., \alpha_n, \alpha_{n+1})$ iff it accepts $(\alpha_1, \alpha_2, ..., \alpha_n)$ and

$$\mid q_{\alpha_n} - q_{\alpha_{n+1}} \mid < \frac{1}{2^{n+1}}$$

$\Gamma_{\Re_{Br}}$ is defined by

$$(\alpha_1, \alpha_2, ..., \alpha_n) \mapsto q_{\alpha_n}.$$

$\Lambda_{\Re_{Br}}$ guarantees the extension of any $\Lambda_{\Re_{Br}}$-admitted sequence $(\alpha_1, \alpha_2, ..., \alpha_n)$, since there always exists rational q such that,

$$q_{\alpha_n} - \frac{1}{2^{n+1}} < q < q_{\alpha_n} + \frac{1}{2^{n+1}}.$$

But q is a q_k, for some k, so $(\alpha_1, \alpha_2, ..., \alpha_n, k)$ is $\Lambda_{\Re}$-admitted. Of course, this q is not unique, so the extension of $(\alpha_1, \alpha_2, ..., \alpha_n)$ is not absolutely determined by $\Lambda_{\Re_{Br}}$. So, Λ_M is, generally, a *non absolutely deterministic law.* Through $\Gamma_{\Re_{Br}}$ the sequence $(q_{\alpha_1}, q_{\alpha_2}, ..., q_{\alpha_n}, ...)$ determines an *intuitionistic real number.*
Two real numbers α, β are *equal*, $\alpha \approx \beta$ if the following condition is satisfied:

$$\alpha \approx \beta \Leftrightarrow |q_{\alpha(n)} - q_{\beta(n)}| < \frac{1}{2^{n-1}}, \quad \forall n \in \omega.$$

Therefore, $(q_{\alpha_1}, q_{\alpha_2}, ..., q_{\alpha_n}, ...)$ is a representative of an intuitionistic real number, which is actually the species of real numbers equal to a representative. I.e., the intuitionistic continuum is the species of the species of real numbers i.e., it is a species of second

order. By that way, the intuitionistic continuum is a *holistic* continuum which *generates* its points, while the classical continuum is an *atomistic* continuum *generated* by its points as their sum (set).
This seemingly strange way to introduce a concept of a set of sequences is justified by Brouwer's need to avoid interpreting the intuitionistic set as a set-box. Choice sequences do not belong to a set, only the spread law holds them together. That's why Brouwer replaced his initial term "Menge" (i.e., set in German) by the new term "spreiding", in English spread, in his 1927 notes. The first books on intuitionism respected the above careful distinctions of terms (see e.g., [Heyting 1966], and less [Beth 1959]), while later presentations use the standard set-theoretical terminology (see e.g., [Troelstra, van Dalen 1988], or [Bridges, Richman 1987]). As Heyting remarks[57]:

> [A spread is not the sum of its elements (this statement is meaningless unless spreads are regarded as existing in themselves). Rather, a spread is identified with its defining rules.]

Note that spread generates new objects, while species hold together already constructed ones.
An equivalent description of the unit interval of intuitionistic real numbers is the following spread.
(V) Let n, k are natural numbers and $\Delta_{n,k}$ is the following closed interval of rational numbers:

$$\Delta_{n,k} = [\frac{n}{2^{k+1}}, \frac{n+2}{2^{k+1}}],$$

where $2 \leq n+2 \leq 2^{k+1}$, hence, $\frac{2}{2^{k+1}} \leq \frac{n+2}{2^{k+1}} \leq \frac{2^{k+1}}{2^{k+1}}$ i.e.,

$$\frac{1}{2^k} \leq \frac{n+2}{2^{k+1}} \leq 1.$$

The left end of $\Delta_{n,k}$, $\frac{n}{2^{k+1}}$ is ≥ 0, and it is 0, only if $n = 0$. The right end of $\Delta_{n,k}$, $\frac{n+2}{2^{k+1}}$, is ≤ 1 and it is 1, only if $n = 2(2^k - 1)$.
Intervals $\Delta_{n,k}$ are obviously countable and let $\Delta_1, \Delta_2, ..., \Delta_n, ...$, a fixed enumeration of them. We define the spread $\Delta[0,1]$ through the spread law $\Lambda_{\Delta[0,1]}$:
(i) Each 1-sequence is accepted.
(ii) $(\alpha_1, \alpha_2, ..., \alpha_n, \alpha_{n+1})$ is $\Lambda_{\Delta[0,1]}$-extension of the $\Lambda_{\Delta[0,1]}$-accepted sequence $(\alpha_1, \alpha_2, ..., \alpha_n)$ iff

$$\Delta_{\alpha_{n+1}} \prec \Delta_{\alpha_n},$$

i.e., interval $\Delta_{\alpha_{n+1}}$ is a subspecies of Δ_{α_n}.
The complementary law $\Gamma_{\Delta[0,1]}$ is the following:

$$\Gamma_{\Delta[0,1]} : \quad (\alpha_1, \alpha_2, ..., \alpha_n) \mapsto \Delta_{\alpha_n}.$$

The choice sequences of the spread $\Delta[0,1]$ is the intuitionistic interval $[0,1]$, and if we define $\Delta_{n,k}$ such that $2\alpha \leq n+2 \leq \beta 2^{k+1}$ we would determine $\Delta[\alpha, \beta]$, the **intuitionistic closed interval** $[\alpha, \beta]$. In that way an intuitionistic real number is described as a sequence of nested intervals of rational numbers.

[57] In [Heyting 1931].

In the definition of the spread of reals a representative $(q_{\alpha_1}, q_{\alpha_2}, ..., q_{\alpha_n}, ...)$ of an intuitionistic real number is actually a Cauchy sequence of rationals. But the intuitionistic continuum is not the "collection" of all law-like given Cauchy sequences of rationals. This is, roughly, the continuum of French semi-intuitionists Borel and Lebesgue, which is countable, since there are only countable laws determining a sequence and it is known as the **reduced continuum**. Such a collection though, is not an intuitionistic object, since there is no simple way to construct the concept of a law-like sequence. The expression "the set of all law-like given Cauchy sequences of rationals" is, according even to Brouwer's early period, just a linguistic expression. So, what matters from the intuitionistic point of view is the concept of real number generator, which is an element or a point of the body of $\Re$. It was through the concept of spread that mature Brouwer found the way to refer to the continuum as a totality, without using the pathological concept of set.
A **real number generator** is an intuitionistic Cauchy sequence (q_n) of rational numbers i.e., for each natural k a natural n_0 *can be found* such that $|q_n - q_m| < \frac{1}{k}$, for each $n, m \geq n_0$. Someone could argue that intuitionistic points are just like points of the classical continuum. Next proposition says that this is not the case.

Proposition 5.1: There is a classical Cauchy sequence, which cannot be accepted as an intuitionistic Cauchy sequence i.e., as a real number generator.

Proof: Following [Dummett 2000] p.26, let (q_n) be defined by

$$q_n = \begin{cases} 1 & \text{, if } 2n+1 \text{ is the first perfect odd number} \\ 2^{-n} & \text{, otherwise} \end{cases}$$

Until now we do not know if there is a perfect odd number (i.e., the sum of its divisors equals its double). Classically, if there is such a perfect odd number, then (q_n) is finally 2^{-n}, while if there is no such number, then (q_n) equals 2^{-n}. In both cases (q_n) is a Cauchy sequence.
Intuitionistically though, if (q_n) was a real number generator, we would have found a natural number n_1 such that $|q_m - q_{n_1}| < \frac{1}{2}$, for each $m \geq n_1$. But then, no q_m could be 1, since $|1 - 2^{-n_1}| > \frac{1}{2}$ i.e., we would know that there is no perfect odd number, which contradicts our lack of this knowledge.$\diamond$

Spreads behave differently from species with respect to intersection or complement operation. Beth (in [Beth 1959] p.425) says that these limitations of spreads made Brouwer to introduce the closer to classical set concept of species. But, in our view, this is not the case and the concept of species is not close at all to the classical set.
If M_1 and M_2 are spreads, then the spread $M_1 \vee M_2$ is easy to define, while $M_1 \wedge M_2$ or M' are not.
If Λ_{M_1} and Λ_{M_2} are the respective spread laws we could define the following law Λ_M:
(i) If $(\alpha_0, \alpha_1, ..., \alpha_k)$ Λ_{M_1}-accepted, then Λ_{M_1} is applied.
(ii) If $(\alpha_0, \alpha_1, ..., \alpha_k)$ Λ_{M_2}-accepted, then Λ_{M_2} is applied.
Therefore, if $(\alpha_0, \alpha_1, ..., \alpha_k)$ Λ_{M_1}, Λ_{M_2}-accepted, then both Λ_{M_1} and Λ_{M_2} are applied.
It is possible though, that a node $(\alpha_0, \alpha_1, ..., \alpha_k)$, is Λ_{M_1} and Λ_{M_2}-accepted, while no extension of it is also Λ_{M_1} and Λ_{M_2}-accepted. Hence, this node cannot be extended as the definition of a spread demands.
Brouwer (in [Brouwer 1923], p.337 of the English translation) uses a fleeing property

to define later two spreads the intersection of which cannot be a spread. A **fleeing property** (fliehende Eigenschaft) is a property, e.g., on natural numbers, $A(n)$, for which the following hold:

(i) $(\forall n)(A(n) \ \vee \ \neg A(n))$.

(ii) We cannot neither prove $\exists n A(n)$ nor $(\forall n)\neg A(n)$.

I.e., while we do not find n, such that $A(n)$, a proof of $(\forall n)\neg A(n)$ escapes. E.g., consider $A(n)$ to be

$A(n)$: the first n elements of the decimal expansion of π contain the sequence 01234567890123456789.

Brouwer (see [van Stigt 1990] p.346) defines M_1 as the spread generating only the zero sequence $\overline{0}$ and M_2 generating only one sequence by the following law:

$$\alpha(n) = \begin{cases} 1 & \text{, if } A(n) \\ 0 & \text{, if } \neg A(n) \end{cases}$$

If the intersection of M_1 and M_2 was a spread, then we must know the law $\Lambda_{M_1 \wedge M_2}$ generating its choice sequences. At no point though, of the generation of α in M_2 we know if α is $\overline{0}$ or not. Thus, we cannot tell if their intersection is M_1 or the empty spread. So, there is no $\Lambda_{M_1 \wedge M_2}$, since a node of length 1 cannot be determined.

We see that a spread is a very general mechanism of generation of sequences, which may depend on our knowledge of a solution of a mathematical problem, causing a lack of knowledge, regarding its behavior. This is the most peculiar characteristic of the spread concept.

Also, the expected law of the complement M':

$(\alpha_0, \alpha_1, ..., \alpha_k)$ is accepted iff $(\alpha_0, \alpha_1, ..., \alpha_k)$ is not Λ_M-accepted,

determines sequences outside the body of M, but it is possible that a finite sequence $(\alpha_0, \alpha_1, ..., \alpha_k)$ is not Λ_M-accepted, while an ancestor of it is, violating condition (ii) of the spread definition.

If we use spread M_2 of the previous counterexample, supposing that M_2' is also a spread, then M_2' must not generate the sequence of M_2 i.e., there must be some k such that $(\alpha_0, \alpha_1, ..., \alpha_k)$ is M_2-accepted, but not M_2'-accepted. Since we do not know some k such that $A(k)$, $(\alpha_0, \alpha_1, ..., \alpha_k)$ must be $(0, 0, ..., 0)$ i.e., $\overline{0}$ is not in $[M_2']$, which we cannot know since A is fleeing.

Another use of a fleeing property is in the following proposition:

Proposition 5.2: It is not intuitionistically accepted that a spread is either the empty spread or a non-empty spread.

Proof: Let M be the spread which generates the constant sequence $\overline{n}$, where n is the first natural number satisfying a fleeing property A. Since we cannot find such an n we cannot say that M is non-empty, and since we cannot show $\forall n \neg A(n)$, we cannot say that M is the empty spread.$\diamond$

If it is impossible that M is empty, then this does not mean intuitionistically that we know a sequence of M i.e., the following implication

$$\neg\neg(\exists \alpha \in [M]) \Rightarrow \exists \alpha \in M$$

is not true.
Proposition A.4 of the Appendix shows that spreads, classically interpreted, correspond to the closed subsets of Baire space.

A *subspread* K of M, $K \preceq M$, is a spread such that, if $(\alpha_1, \alpha_2, ..., \alpha_n)$ is Λ_K-accepted, then it is also Λ_M-accepted.
As it is expected from the language of trees, a spread M is called *splitting* iff $(\forall \alpha \in [M])(\forall N)(\exists \beta \in [M])(\exists K > N)(N_\beta = N_\alpha \ \wedge \ K_\beta \neq K_\alpha)$, i.e., if each finite M-sequence splits at some moment of its evolution into two different sequences.
A **fan** F is a finitely branching spread i.e., each finite Λ_F-admitted sequence can be extended only by finitely many naturals. A *subfan* T of F, $T \preceq F$, is a subspread of F. If the universal law applies only to $0-1$ sequences, then we take the fan 2^ω, which corresponds to the classical Cantor space $\mathcal{C}$. All branches of an intuitionistic fan are considered infinite. In Paragraph 12 we show that intuitionistic $[\alpha, \beta]$ is also a fan, a result necessary to Brouwer's proof of Uniform Continuity theorem.
The basic function of the spread concept is the description of a holistic and uncountable continuum (see Proposition 8.2) without the use of absolute infinity. The cost of this delicate function is that its generating choice sequences are incomplete, on-going objects on which classical logic cannot be applied. In a sense, intuitionistic logic is the logic of incomplete, on-going objects. Next paragraphs show how the study of intuitionistic spreads deviates from the study of classical spreads (the closed subsets of Baire space) because of the incomplete nature of the infinite sequences of intuitionistic spreads.

6. Brouwer's continuity principle as a result of a definition and not as an axiom. Brouwer conceived Continuity Principle (CP) in relation to Cantor's diagonal argument. He lectured on it even from 1915/16, though he introduced it in [Brouwer 1918] p.13. CP is not classically true, but its intuitionistic truth derives from the study of sequences on a different kind of continuum. CP is formulated as follows:

Continuity Principle: If ω^ω is the body of the universal spread[58] and $\varphi : \omega^\omega \to \omega$ a function on ω^ω, then for each choice sequence α in ω^ω there is a natural number N, such that, for each sequence β, which shares with α the same N-initial segment, β has the same value under φ with α. In symbols:

$$(CP) \qquad \forall \alpha(\alpha \in \omega^\omega)(\exists N)(\forall \beta, N_\beta = N_\alpha \Rightarrow \varphi(\beta) = \varphi(\alpha)),$$

where N_α, is the N-initial segment of α. As we show in the Appendix, CP expresses the continuity of φ if the species ω^ω is interpreted classically as a set. I.e.,

If a function $\varphi : \omega^\omega \to \omega$ is interpreted classically, then it is always continuous.

While, classically, a function $\varphi : \mathcal{N} \to \mathbb{N}$ satisfying CP is continuous, the intuitionistic principle CP asserts that all $\varphi : \omega^\omega \to \omega$ are continuous. The clash though, is only apparent. BIA and classical analysis behave differently on objects which have only a common name. As Feferman notes (in [Feferman 1997c] p.222) regarding Brouwer's Uniform Continuity theorem[59],

> [This (Brouwer's Uniform Continuity theorem), on the face of it, is in direct contradiction to classical mathematics, but once it is understood that

[58]For simplicity, we identify ω^ω with $[\omega^\omega]$.
[59]According to it a real function on $[a, b]$ is uniformly continuous (see Paragraph 12).

> Brouwer's theorem must be explained differently via the intuitionistic interpretation of the notions involved, an actual contradiction is avoided. Perhaps if different terminology had been used, classical mathematicians would not have found the intuitionistic redevelopment of analysis so off-putting, if not downright puzzling.]

The above comment suits CP too. Brouwer's justification of CP reflects the intuitionistic meaning of a function $\varphi : \omega^\omega \to \omega$, which is different from a classical function $\varphi : \mathcal{N} \to \mathbb{N}$.
The standard attitude towards CP is to treat it as an axiom after an intuitive justification of it. For example CP is found as Brouwer's Principle for numbers in [Kleene, Vesley 1965], or as the $WC - \mathbb{N}$ axiom in [Troelstra, van Dalen 1988a].
The **standard intuitive justification of CP** is as follows:

Function φ is a kind of rule, which corresponds to each choice sequence α a unique natural number. Sequence α though, is an on-going object of which we always know an initial segment. Thus, the value $\varphi(\alpha)$ must depend on some initial segment N_α of α. The way α grows after N_α is irrelevant to the value of α under φ. So, each sequence β with the same N-initial segment to that of α will have the same value under φ with that of α.

Brouwer himself considered CP as obviously true and for that reason he never bothered justifying it more, using it freely. In order though to fully establish CP we need to say more.
Treatment of CP as an evident truth gave CP gradually the character of "a natural axiom, borne out by experience"[60]. This character though, is not consistent with BIA's constructive character. CP guarantees, given a function φ, for every choice sequence α, the existence of an object, that of N, for which it does not provide a method of constructing it. Even if someone accepts the above standard justification of CP, CP, treated as an axiom, is constructively questionable.
There is another, more serious reason within BIA for not treating CP as an axiom. Axiomatic definition of a concept is not in Brouwer's spirit. To understand the concept of a function $\varphi : \omega^\omega \to \omega$ through the axioms in which this concept is found is an approach that Brouwer confronted from his youth. That this was not Brouwer's way is clear from his attitude towards Fan theorem. This too, or Bar theorem, can be considered as an axiom, but Brouwer tried to prove it and he never considered it as an axiom.
Hence, if we reject the axiomatic understanding of a concept, the only way to start understanding CP is to clarify the concept of a function $\varphi : \omega^\omega \to \omega$ from the intuitionistic point of view. Before we assert anything on functions $\varphi : \omega^\omega \to \omega$ we must say how we understand them. So, we need to define such a function. This attitude is a fundamental element in our reconstruction of Brouwer's analysis. A short description of it is:

"*BIA contains only definitions of concepts and not axioms.*"

This a fundamental characteristic of a self-interpreted mathematical theory and BIA is reconstructed as such a theory in [Petrakis 2010].

[60] This is a phrase of Veldman in [Veldman 1999] p.287.

A classical function $\varphi : \mathcal{N} \to \mathbb{N}$ can be interpreted as an automaton, with input a sequence α and output the natural number $\varphi(\alpha)$.

$$\xrightarrow{\alpha\in\mathcal{N}} \boxed{\quad\varphi\quad} \xrightarrow{\varphi(\alpha)}$$

Classical sequence α is a completed object in W and its value $\varphi(\alpha)$ is independent of our knowledge of how φ operated on α. An intuitionistic $\varphi : \omega^\omega \to \omega$ will be a special case of a function $\varphi : A \to \omega$, where A is a species of choice sequences. A common element of all these definitions is that the operation of φ depends on the way the elements of A are defined.
In the case of a $\varphi : \omega^\omega \to \omega$ the only thing we know of a sequence α in ω^ω is an initial segment of α.

An **intuitionistic ω^ω-function**, $\varphi : \omega^\omega \to \omega$, is a law[61] Λ_φ, such that:
(i) Λ_φ corresponds an ω^ω-sequence α to a unique natural number $\varphi(\alpha)$, based on an initial segment of α of length N, N_α, for some N, or on any extension of it. We call any such node a *critical node* for φ.
(ii) Λ_φ decides effectively if an initial segment M_α of α is a critical node for φ or not. If not, then there is no output (and conversely), while it gives the same output for all extensions of a critical node[62].

This definition is completely natural, since α is an on-going object and its value must be determined some time in the course of its 'becoming', if we want $\varphi(\alpha)$ to depend on our knowledge of α. It is this on-going character of intuitionistic sequences and the aforementioned identification between ontology and epistemology in intuitionism which *force* the above definition.
If M is an arbitrary spread, an **intuitionistic M-function** $\varphi : M \to \omega$, is defined likewise.
In previous paragraph we saw that an intuitionistic sequence was identified with a spread choice sequence. In complete analogy, a spread function φ is identified with a function φ^* on finite nodes[63].
In analogy to the classical automaton, intuitionistic φ is represented as follows:

$$\xrightarrow{(\alpha_0,\alpha_1,\ldots,\alpha_{N_\alpha})} \boxed{\quad\varphi\quad} \xrightarrow{\varphi(\alpha)}$$

Actually, φ is determined by a function φ^* on the finite $\omega^{<\omega}$-sequences. The existence of Λ_φ is equivalent to the existence of such a φ^* satisfying:
(i') For each α, there is some N_α such that,

$$\varphi(\alpha) = \varphi^*(N_\alpha).$$

(ii') φ^* decides effectively if an initial segment M_α of α is a critical node or not. Again no output means that M_α is not critical and if M_α is critical and $N_\alpha \succ M_\alpha$, then

[61] In Brouwer's words: "...by a function...we understand a law..." ([Brouwer 1927] p.458.

[62] There are many, more or less, equivalent formulations of the same concept.

[63] Epple (in [Epple 1997]) also introduces a spread function through a definition without though, elaborating on the consequences, philosophical and technical, of such an attitude.

$\varphi^*(N_\alpha) = \varphi^*(M_\alpha)$.
Λ_φ is actually Λ_{φ^*} and the automaton scheme becomes:

$$\xrightarrow{(\alpha_0,\alpha_1,\ldots,\alpha_{N_\alpha})} \boxed{\;\varphi^*\;} \xrightarrow{\varphi(\alpha)}$$

and we say that φ^* computes φ.

Proof of Continuity Principle: If β is a sequence such that $N_\beta = N_\alpha$, then φ corresponds β to $\varphi(\alpha)$, since φ, by its definition, is activated only by N_α. I.e.,

$$\varphi(\beta) = \varphi^*(N_\beta) = \varphi^*(N_\alpha) = \varphi(\alpha). \quad \diamond$$

Within our reconstruction of BIA, CP does not certify the existence of a natural number without indicating a way of finding it. The existence of N_α, for each α, is part of the way an intuitionistic function works and this is an information that such a function carries with itself.
CP is classically false[64], since the following function

$$\varphi(\alpha) = \begin{cases} 1 & \text{, if } \alpha \neq \overline{0} \\ 0 & \text{, if } \alpha = \overline{0} \end{cases}$$

where $\overline{0}$ is the constant sequence 0, does not satisfy CP. But the above φ is not an intuitionistic function since, if it were, it would correspond $\overline{0}$ to 0, based on a $N_{\overline{0}}$, for some N. Consequently, sequences other than $\overline{0}$ would also correspond to 0 through φ.
Working exactly like the ω^ω-case, we get the continuity principle CP(M) for arbitrary spread M.

$$CP(M) \qquad \forall\alpha(\alpha \in [M])(\exists N)(\forall\beta, N_\beta = N_\alpha \Rightarrow \varphi(\beta) = \varphi(\alpha)).$$

As we show in Proposition 9.3, CP(M) is a direct consequence of CP.
In BIA Brouwer studied only sequences generated within some spread M, thus CP(M) holds for them. Later studies of choice sequences extended the way a choice sequence is born and the validity of CP was a matter of examination. If a function φ is defined on such sequences α, then the information needed for the action of φ is larger than a finite initial segment of α. The study of such sequences had not always clear intuitionistic motivation. We may also though, preserve the definitional approach to such extended situations.
Let A a well-constructed species of sequences α. We define an *intuitionistic A-finction* $\varphi : A \to \omega$, or A-function φ_A, a correspondence law, for which the following hold:

(i) φ_A gives the unique value $\varphi_A(\alpha)$ to α, relying on a finite amount of information $\Pi(\alpha)$ concerning α as an input. Information $\Pi(\alpha)$ is formulated in a way compatible to the way A is defined.
(ii) φ_A answers effectively the question whether a finite amount of information $\Pi'(\alpha)$ regarding α, ($\Pi(\alpha)$ and $\Pi'(\alpha)$ are analogously formulated) as an input activates $\varphi(\alpha)$.

[64] In Kleene's system CP is the only formal axiom which separates his system of intuitionistic analysis (FIM) from classical analysis. In our view though, this single formal difference does not grasp the difference between the classical and the intuitionistic framework.

If there is such an output for $\Pi'(\alpha)$, it is always $\varphi(\alpha)$. Especially, if $\Pi'(\alpha)$ contains $\Pi(\alpha)$, then the output is always $\varphi(\alpha)$.

Schematically, φ_A is of the form:

$$\xrightarrow{\Pi(\alpha)} \boxed{\;\varphi\;} \xrightarrow{\varphi(\alpha)}$$

Thus, an A-function is determined by the input sequences α, the kind of the finite information $\Pi(\alpha)$ which activates its mechanism, and the values $\varphi(\alpha)$. All these three elements of φ_A must be compatible to each other.
The continuity principle corresponding to A-functions expresses the fact that for each sequence α of A there is a finite amount of information $\Pi(\alpha)$ for which each A-sequence β accompanied with finite information $\Pi(\beta)$ "equal" to $\Pi(\alpha)$, has the same value under φ with that of α. I.e.,

$$(CPA) \qquad \forall\alpha(\alpha \in A)(\exists\Pi(\alpha))(\forall\beta, \Pi(\beta) = \Pi(\alpha) \Rightarrow \varphi(\beta) = \varphi(\alpha)).$$

Obviously, equality $\Pi(\beta) = \Pi(\alpha)$ is defined with respect to the nature of A.

Proof of CPA: If β is a sequence with $\Pi(\beta) = \Pi(\alpha)$, then, since by (i), φ is activated by $\Pi(\alpha)$, and produces $\varphi(\alpha)$ as an output, then φ corresponds $\varphi(\alpha)$ to β too.$\diamond$

Again, CPA is a simple consequence of the way an A-function is defined.
Of course, there is no a priori reason that CPA leads to a CP of an extensional character, since there is no a priori reason that the needed information $\Pi(\alpha)$ to be contained to an initial segment of α. Hence, a question found in post-Brouwer literature is, if it is possible the following extensional continuity principle could hold:

$$(CPE) \qquad \forall\alpha(\alpha \in A)(\exists N)(\forall\beta, N_\beta = N_\alpha \Rightarrow \varphi(\beta) = \varphi(\alpha)).$$

Hence, the question is:

$$(CPA) \overset{?}{\Rightarrow} (CPE).$$

van Atten and van Dalen, in [van Atten, van Dalen 2002], trying to justify CPE, without though considering all the above definitions, provide some examples which are worth discussing under the light of them.

The first example, formulated in our language, is the following:
Function φ corresponds to each sequence its $100th$ term. For a sequence α its four first terms are introduced together with the information that α is constant after its fourth term. If β is a sequence with the same 4-segment, then φ does not send β to the same value with α, since β may evolve in a different way. Thus, the extensional information which activates $\varphi(\alpha)$ does not activate $\varphi(\beta)$.
van Atten and van Dalen say that this example suggests a violation of CP, since φ does not behave like a universal φ. This violation though, is explained by the fact that the information on α is larger than any of its initial segments. I.e., CP is violated but CPA is not, since the information $\Pi(\beta)$ on β is strictly less than $\Pi(\alpha)$. The same example couldn't bother also one who believed in CPE, since all the information which accompanies the 4-segment of α can take an extensional form. $\Pi(\alpha)$ contains the information that all 100 terms of α are $\underbrace{(\alpha(0), \alpha(1), \alpha(2), \alpha(3), \alpha(3), ..., \alpha(3))}_{100}$, therefore

any sequence with the same 100-segment has the same value under φ.
Hence, it makes sense to argue that:

A necessary and sufficient for $(CPA) \Rightarrow (CPE)$ to hold is that the information $\Pi(\alpha)$, for each input α to be equated to an initial segment of α.

In the second example of van Atten and van Dalen A is the species of all finally constant sequences. For each α in A the activating $\Pi(\alpha)$ is an initial segment of α containing the constant value and the output is that constant value. Obviously, if we introduce an N_α without the information that one of the constant term is contained in N_α, then $\varphi(\alpha)$ is not activated. Moreover, the activating information $\Pi(\alpha)$ is not equivalent to any of the initial segments of α, since $\Pi(\alpha)$ contains complete knowledge of α. Each plane segment N_α though, does not contain the information that α is constant. Obviously φ satisfies CPA, but it does not satisfy CPE. Thus $(CPA) \Rightarrow (CPE)$ cannot hold in general, since there are activating information which are not contained in any initial segment of a sequence. A strict finitist though, i.e., a man denying even the potential kind of infinity, wouldn't consider the above information $\Pi(\alpha)$ on α as finite, but that seems to us too narrow point of view.
Such generalized functions corresponding to general kinds of species of sequences were studied in post-Brouwer literature[65]. The naturally arising question[66], is to find those species of sequences for which $(CPA) \Rightarrow (CPE)$ holds.

7. Immediate consequences of the Continuity Principle. The first application of CP in [Brouwer 1918] was the proof of uncountability of Baire space, independently from Cantor's diagonal argument. Brouwer reaches uncountability through CP in the most direct way. Of course, uncountability is at first a negatively defined concept and any proof of

$$N(a) : N \textit{ is a negatively defined concept},$$

has to be a reductio ad absurdum proof.

Proposition 7.1: The universal spread ω^ω is uncountable i.e., there is no universal intuitionistic function $\varphi : \omega^\omega \overset{1-1}{\to} \omega$.

Proof: If $\varphi : \omega^\omega \to \omega$, then, by CP, $\exists N$ such that $\forall \beta, N_\beta = N_\alpha \Rightarrow \varphi(\beta) = \varphi(\alpha)$. Hence, condition $\varphi(\beta) = \varphi(\alpha)$ does not entail $\alpha = \beta$.$\diamond$

Thus, for any fixed function φ and sequence α, there is a sequence $\beta \neq \alpha$ such that $\varphi(\beta) = \varphi(\alpha)$.
Hence ω^ω is not equipollent to any countable species of natural numbers, since a function $\varphi : \omega^\omega \to \omega$ cannot be defined, while within Cantor's proof, ω^ω is not countable since each function $f : \omega \to \omega^\omega$ does not exhaust ω^ω. Cantor's proof is intuitionistically a legitimate one, although Brouwer's proof stems immediately from his concept of an intuitionistic function $\varphi : \omega^\omega \to \omega$.

Proposition 7.2: The intuitionistic continuum $\Re_{Br}$ is uncountable.

Proof: Suppose $\varphi : \Re_{Br} \overset{1-1}{\to} \omega$. If $(q_{\alpha_0}, q_{\alpha_1}, ..., q_{\alpha_n}, ...)$ determines an irrational number and $(q_{\alpha_0}, q_{\alpha_1}, ..., q_{\alpha_n})$ is a critical segment to φ^*, then sequence $(q_{\alpha_0}, q_{\alpha_1}, ..., q_{\alpha_n}, q_{\alpha_n}, q_{\alpha_n}, ...)$ the dress of $(\alpha_0, \alpha_1, ..., \alpha_n, \alpha_n, \alpha_n, ...)$, which determines the rational number q_{α_n}, has

[65] See e.g., [Troelstra 1977] and [Troelstra, van Dalen 1988b].
[66] See e.g., [van Atten, van Dalen 2002].

the same value with $(q_{\alpha_0}, q_{\alpha_1}, ..., q_{\alpha_n}, ...)$ under φ^*, which is absurd, since $[q_{\alpha_n}]$ is rational and $[(q_{\alpha_0}, q_{\alpha_1}, ..., q_{\alpha_n}, ...)]$ is irrational. $\diamond$

Of course, there are spreads generating a finite number of choice sequences, therefore, with a countable body (note that countability is a positively defined concept). It is interesting to determine the uncountable spreads generalizing the previous two proofs. Each initial segment of a sequence in ω^ω or $\Re_{Br}$ splits at some point. This ensures uncountability. As we show in Proposition A.7, a splitting spread in Baire space has no isolated points. The following result is an immediate generalization of the previous uncountability facts.

Proposition 7.3: A non-empty splitting spread M is uncountable.

Proposition 7.3 is the intuitionistic analogue to the following classical proposition, which generalizes Cantor's result of the uncountability of 2^ω. Since its classical proof is interesting from the intuitionistic point of view we give it next.

Proposition 7.4 (Generalized Cantor's theorem): A non-empty perfect (i.e., closed (classical spread) and splitting) set M of $\mathcal{N}$ has the cardinality of the continuum.

Proof: Classically, $\mathcal{C}$ is proven to be uncountable through Cantor's diagonal argument, and, since it is equipollent to $\mathcal{P}(\mathbb{N})$, it has the cardinality of the continuum. Obviously, $\mathcal{C}$ is a perfect set. In order to show that a non-empty perfect set M of $\mathcal{N}$ has the cardinality of the continuum it suffices to show that it contains a copy of $\mathcal{C}$, i.e., that $\mathcal{C}$ is embedded to M.
Since M is splitting, we define functions $A, \Delta : M^{<\omega} \to M^{<\omega}$, with $A(\xi) = b$ and $\Delta(\xi) = c$, where b, c are incomparable extension nodes of ξ (that can be found effectively).
Classically, at this point a choice principle is used, which is intuitionistically though accepted, by the intuitionistic interpretation of existence. I.e., the existence for each node ξ of nodes b, c is guaranteed by the spread law Λ_M.
Through A, Δ the following $\varphi : 2^{<\omega} \to M^{<\omega}$ is defined:

(I) $\varphi(<>) = <>$.
(II) $\varphi(\xi * 0) = A(\varphi(\xi))$.
(III) $\varphi(\xi * 1) = \Delta(\varphi(\xi))$.

This recursive definition is intuitionistically accepted. Through A, Δ φ corresponds to each finite $0,1$-sequence a node of M. $0'$s correspond to a left, A, split, while $1''$s to a right, Δ, split. E,g,.

$$(1, 0, 0, 1, 0) \overset{\varphi}{\mapsto} (A\Delta AA\Delta(<>)).$$

The construction of φ resembles the construction of K_ξ in the topological characterization of $\mathcal{C}$ (Proposition A.8).
Clearly, φ satisfies the following properties:

(i) $l(\xi) \leq l(\varphi(\xi))$
(ii) $\varphi(\xi) \preceq \varphi(\xi * 0)$, since $\xi \preceq A(\xi)$.
(iii) $\varphi(\xi) \preceq \varphi(\xi * 1)$, since $\xi \preceq \Delta(\xi)$.
(iv) $\varphi(\xi) \preceq \varphi(\xi * \zeta)$, by (ii) and (iii).
(v) $\xi \preceq \zeta \Rightarrow \varphi(\xi) \preceq \varphi(\zeta)$, by (iv).
(vi) $\xi \bowtie \zeta \Rightarrow \varphi(\xi) \bowtie \varphi(\zeta)$.

(vi) is justified by the fact that the split of the common segment of ξ and ζ at some level i leads to a split of the image of the common segment under φ at some level j.
Through φ we define $\tilde{\varphi} : \mathcal{C} \to M$ by:

$$\alpha \overset{\tilde{\varphi}}{\mapsto} \lim_N \varphi(N_\alpha) = \sup\{\varphi(N_\alpha), N \in \omega\}.$$

$\tilde{\varphi}$ is $1-1$, since, if $\alpha \neq \beta$, then $N_\alpha \neq N_\beta$, for some N, i.e., $N_\alpha \bowtie N_\beta$, thus, by (vi), $\varphi(N_\alpha) \bowtie \varphi(N_\beta)$, which amounts to $\tilde{\varphi}(\alpha) \neq \tilde{\varphi}(\beta)$.
Actually, $\tilde{\varphi}$ is a continuous function computed by φ.$\diamond$

Just as the proof of Cantor's theorem is intuitionistically accepted, although uncountability is interpreted as sequential inexhaustibility, the above proof of generalized Cantor's theorem is intuitionistically accepted, although the proof of Proposition 7.3 is more direct expressing intuitionistically the content of generalized Cantor's theorem. Thus, the following are in complete analogy:

$$\frac{\neg(\exists e : \omega^\omega \to \omega)}{Cantor} = \frac{\neg(\exists e : M \to \omega)}{Gen.Cantor}.$$

In intuitionistic descriptive set theory the question which spreads cannot be embedded to ω i.e., which spreads behave like splitting spreads, is studied (see [Petrakis 2010]).

Finally, we discuss two related propositions. The first, in [van Atten, van Dalen 2002] p.340, translated though, in our language, is the following:

Proposition 7.5: Assume the creating subject generates choice sequences as individual objects, and can therefore enumerate the sequences generated so far. Then, CP does not hold.

The "proof" of this proposition is similar to the proof of Proposition 7.1. The hypothesis of enumeration of choice sequences by the creating subject (CS) obviously contradicts CP. This proposition, together with the following one, are treated by van Atten and van Dalen as arguments against the universal truth of CP. Within our reconstruction of the intuitionistic function though, this is not the case.
One way, not the only one, that CS enumerates choice sequences is the following: First he determines $\alpha_1(0)$, secondly $\alpha_1(1)$ and $\alpha_2(0)$, thirdly $\alpha_1(2)$, $\alpha_2(1)$ and $\alpha_3(0)$ and so on. This enumeration though, of the choice sequences does not result from a universal φ, but it is constructed in parallel to gradually formed sequences. The whole structure of Proposition 7.5 is the mixture of two different frameworks regarding the concept of function. CP is the result of a certain understanding of a function, with respect to which a universal function is a mechanism activated by finite nodes and at the same time it is independent from them. The above enumeration of the CS though, is an incomplete, on-going object too, absolutely dependent on the choice sequences it enumerates. Of course, this incomplete object violates CP, but this fact cannot affect the validity of CP relative to an intuitionistic function, a complete object, exactly like the spread law, which is defined independently too from its generating choice sequences.
van Atten and van Dalen (in [van Atten, van Dalen 2002] pp.340-41) prove the following variation of previous proposition.

Proposition 7.6: If $(\alpha_n)_n$ is an enumeration of choice sequences, then a functional is defined through $(\alpha_n)_n$ violating CP.

The above result is, in our opinion, of no special importance regarding our CP, since there is no indication to how the supposed enumeration is constructed. I.e., a violation of CP is generated by a hypothesis with no constructive content.
The above two propositions result from an axiomatic approach to CP and a linguistic treatment of the function concept, resembling classical mathematics.

8. Brouwer's external notion of real Function. For Brouwer a real Function i.e., a function from a species of real numbers like the unit continuum, where a real number (or point core) α is the species of all reals equal to α, to the species of real numbers i.e., a *real Function* $\Phi : [0,1]_{Br} \to \mathfrak{R}_{Br}$, is[67]

> [a law that, with each of certain point cores of the unit continuum, which will be denoted by ξ and form the "domain of definition" of the function, associates one point core of the linear continuum, which will be denoted by $\eta = \Phi(\xi)$]

Therefore, $\Phi : [0,1]_{Br} \to \mathfrak{R}_{Br}$ is a law Λ_Φ such that

$$\eta \overset{\Lambda_\Phi}{\mapsto} \Phi(\xi).$$

In contrast to his concept of an intuitionistic function $\varphi : \omega^\omega \to \omega$, which was treated (by Brouwer) or defined (by us) internally, Brouwer's concept of real Function[68] is defined by Brouwer externally. A function φ is treated or defined *internally*, since φ is not just a law which sends choice sequences to naturals but the way this correspondence is achieved is an essential part of the concept φ. On the other hand, Φ is defined by Brouwer *externally*, since Φ is just a law which corresponds point cores to point cores without any explication of how this correspondence is achieved. So, there is an essential conceptual difference between Brouwerian concepts φ and Φ, which prevailed also in post-Brouwer presentations of the same concepts. Generally φ is treated internally, either through a definition (e.g., see [Epple 1997]) or, standardly, through the continuity principle (axiom). Functions Φ are treated as laws possessing no internal description of their structure. In this paragraph we discuss this asymmetry of the two concepts and in the next one we present an internally defined concept of intuitionistic Function Φ in complete analogy to φ.
In [Brouwer 1927] Brouwer proved his negative continuity theorem, namely that a hypothesis of discontinuity of a real Function leads to an unacceptable proposition i.e., to a weak counterexample. This result is independent from CP i.e., the argument used does not take into account CP.
A real Function Φ is *positively continuous* at a point core ξ_0 iff for each rational $\varepsilon > 0$, there is a rational a_ε such that

$$|\xi - \xi_0| < a_\varepsilon \Rightarrow |\Phi(\xi) - \Phi(\xi_0)| < \varepsilon.$$

At this point we do not explain all the above terms, something we do soon, when we present Veldman's results.

[67] See [Brouwer 1927] p.458.
[68] We use the term "Function" for a mapping with choice sequences as values and the term "function" for a mapping with naturals as values.

A real Function Φ is *negatively continuous* at a point core ξ_0 iff for each Cauchy sequence $(\xi_n)_n$ of point cores such that $\xi_n \to \xi_0$, then $\Phi(\xi_n) \to \Phi(\xi_0)$ negatively i.e., $\neg\neg[\Phi(\xi_n) \to \Phi(\xi_0)]$.
The following proposition was known to Brouwer since 1918.

Proposition 8.1 (Brouwer's negative continuity theorem (BNCT) 1927): If $\Phi : [0,1]_{Br} \to \mathfrak{R}_{Br}$ is a real Function, then Φ is negatively continuous i.e., it is negatively continuous at each point core ξ_0 of $[0,1]_{Br}$.

Proof: Let $(\xi_n)_n$ be a Cauchy sequence of point cores such that $\xi_n \to \xi_0$, for some point core ξ_0, and suppose $\neg[\Phi(\xi_n) \to \Phi(\xi_0)]$. Thus, without loss of generality, there is a rational $\frac{1}{p}$, where p is a natural, and a sequence of naturals p_n, such that, for each n

$$(*) \quad |\Phi(\xi_{p_n}) - \Phi(\xi_0)| > \frac{1}{p_n},$$

where $p_n > p_{n-1}$, for each n.
Then, a point core ξ_ω of the unit continuum is defined as follows: for each n, the first n steps of the formation of ξ_ω, actually of a representative of ξ_ω, are the same to the corresponding steps of ξ_0. These steps either concern finite families sub-intervals of $[0,1]_{Br}$ or finite sequences of rationals. But at each $(n+1)$-step the creating subject reserves the right to choose for all the rest steps to follow the steps of formation of ξ_{p_n}. That is possible, since $\xi_n \to \xi_0$. In that way a point core of $[0,1]_{Br}$ is gradually formed, but we cannot know beforehand its value $\Phi(\xi_\omega)$, since we cannot know if ξ_ω is actually ξ_0 or some ξ_{p_n}, and because of $(*)$, $\Phi(\xi_0) \neq \Phi(\xi_{p_n})$, for each n. Therefore, we have reached a contradiction, since we had supposed that Φ was a *full* function i.e., with $[0,1]_{Br}$ as its domain of definition, and a point core ξ_ω of $[0,1]_{Br}$ was constructed for which its value under Φ cannot be calculated. If it was, then a decision of the creating subject would be known before it was taken, and that is impossible. We could reach the same impossibility, if instead an unsolved mathematical problem was used in the construction of ξ_ω.$\diamond$

Brouwer included this weak result in his 1927 paper because he believed that his BNCT was suggestive to his UCT, that every full real Function on the unit continuum is uniformly continuous, which presupposes his fan theorem.
BNCT is in a sense an expected result of the external, therefore independent of time, character of a real Function. While the inputs of Φ, the core points of $[0,1]_{Br}$ are on-going objects, being generated in time, Λ_Φ is timeless and pre-existent. In that way it is not strange that a choice sequence is formed in time such that its value under Φ depending on its way of formation cannot be calculated. In our opinion BNCT is the result of the incompatibility between the on-going inputs ξ and the timeless Function law Λ_Φ. This time asymmetry is not found in the case of an intuitionistic function φ, the law Λ_φ of which respects the on-going character of its inputs. Thus, Brouwer's result seems to us philosophically poor, since it is the result of an asymmetrical co-existence of concepts, in the same way results on externally defined φ seemed to us poor in Paragraphs 6 and 7.
Veldman, in [Veldman 1982], proved that CP guarantees the pointwise continuity of a function defined on the spread of canonical real numbers, independently from Brouwer's Uniform Continuity theorem and consequently from Fan theorem. Of course, Brouwer's theorem is much stronger, but Veldman's result is worth mentioning due to its inde-

pendence from BFT and the use of actually the same concept of real Function.
Having fixed an enumeration of rational numbers, a *real number* α (r.n) is an element of ω^ω such that:

$$|q_{\alpha(n)} - q_{\alpha(n+1)}| < \frac{1}{2^{n+1}}, \qquad \forall n \in \omega. \qquad (1)$$

The spread of r.n. $\mathfrak{R}_{Br}$ is determined by the above condition which generates its elements. As we said in Paragraph 5, the equality of two r.n. α, β is defined by

$$\alpha \approx \beta \Leftrightarrow |q_{\alpha(n)} - q_{\beta(n)}| < \frac{1}{2^{n-1}}, \qquad \forall n \in \omega. \qquad (2)$$

A *real Function* $\Phi : \mathfrak{R}_{Br} \to \mathfrak{R}_{Br}$ is a law Λ_Φ which corresponds to each r.n. α a r.n. $\Phi(\alpha)$

$$\alpha \overset{\Lambda_\Phi}{\mapsto} \Phi(\alpha) \quad s.t.,$$

$$\alpha \approx \beta \Rightarrow \Phi(\alpha) \approx \Phi(\beta). \qquad (3)$$

Veldman's definition, is actually Brouwer's, since, due to (3), a point core of the continuum is sent to another point core, and it is also external, since it does not explain how such a correspondence $\alpha \overset{\Lambda_\Phi}{\mapsto} \Phi(\alpha)$ is understood. In our opinion though, the correspondence of infinitely proceeding sequences to other such sequences begs for such an understanding, the same way this understanding was needed in the intuitionistic function φ-case.
The definition of operations between r.n. is straightforward. The sum, for example, is defined by $(\alpha + \beta)(n) = m$, where m is the index of $q_{\alpha(n)} + q_{\beta(n)}$ in the fixed enumeration of rationals i.e.,

$$q_{(\alpha+\beta)(n)} = q_{\alpha(n)} + q_{\beta(n)},$$

where

$$\alpha \mapsto q_{\alpha(n)}$$

is the standard correspondence between a r.n. α and its rational approximation. Naturally,

$$q_{|\alpha(n)|} = |q_{\alpha(n)}|$$

and

$$\alpha < \beta \Leftrightarrow (\forall_n)(q_{\alpha(n)} < q_{\beta(n)}). \qquad (4)$$

Therefore, the composite expression

$$|\alpha - \beta| < \frac{1}{2^k}$$

means that

$$(\forall n) \quad q_{|\alpha-\beta|(n)} = |q_{(\alpha-\beta)(n)}| = |q_{\alpha(n)} - q_{\beta(n)}| < \frac{1}{2^k}. \qquad (5)$$

Under the above understanding we prove the following proposition.

Proposition 8.2 (Veldman 1982): If $\Phi : \mathfrak{R}_{Br} \to \mathfrak{R}_{Br}$ is a real function (in the above sense of Veldman), then Φ is continuous at every point of $\mathfrak{R}_{Br}$ i.e.,

$$(\forall \alpha \in \mathfrak{R}_{Br})(\forall m)(\exists n)(\forall \beta \in \mathfrak{R}_{Br}) \quad |\alpha - \beta| < \frac{1}{2^n} \Rightarrow |\Phi(\alpha) - \Phi(\beta)| < \frac{1}{2^m}. \qquad (6)$$

Proof: We fix a natural number m and we define the following intuitionistic function $\theta : \mathfrak{R}_{Br} \to \omega$

$$\alpha \mapsto \Phi(\alpha)(m+2),$$

for each r.n. α [69]. Fixing a r.n. α and applying CP on θ and the spread $\mathfrak{R}_{Br}$ we get

$$(\exists n)(\forall \gamma)((n_\gamma = n_\alpha) \Rightarrow \Phi(\gamma)(m+2) = \Phi(\alpha)(m+2)). \qquad (7)$$

As a result of (7)

$$|q_{\Phi(\alpha)(m+2+k)} - q_{\Phi(\gamma)(m+2+k)}| < \frac{1}{2^{m+1}}. \qquad (8)$$

To see why (8) is true we check first the case $k = 1$.

$$|q_{\Phi(\alpha)(m+3)} - q_{\Phi(\gamma)(m+3)}| = |q_{\Phi(\alpha)(m+3)} - q_{\Phi(\alpha)(m+2)} + q_{\Phi(\gamma)(m+2)} - q_{\Phi(\gamma)(m+3)}|$$

$$< |q_{\Phi(\alpha)(m+3)} - q_{\Phi(\alpha)(m+2)}| + |q_{\Phi(\gamma)(m+2)} - q_{\Phi(\gamma)(m+3)}| < \frac{2}{2^{m+3}} = \frac{1}{2^{m+2}} < \frac{1}{2^{m+1}},$$

since the triangle inequality holds directly on rationals. Working likewise, if $k = 2$, then

$$|q_{\Phi(\alpha)(m+4)} - q_{\Phi(\gamma)(m+4)}| < (\frac{3}{4})\frac{1}{2^{m+1}},$$

and in the general case

$$|q_{\Phi(\alpha)(m+2+k)} - q_{\Phi(\gamma)(m+2+k)}| < (\frac{2^k - 1}{2^k})\frac{1}{2^{m+1}}.$$

As a consequence of (8)

$$|\Phi(\alpha) - \Phi(\gamma)| < \frac{1}{2^m}, \qquad (9)$$

since

$$|q_{\Phi(\alpha)(n)} - q_{\Phi(\gamma)(n)}| < \frac{1}{2^{m+1}} < \frac{1}{2^m},$$

for $n \geq m+2$.
If β is any r.n. such that $|\alpha - \beta| < \frac{1}{2^n}$, then

$$(\exists \gamma)\ (n_\gamma = n_\alpha\ \wedge\ \beta \approx \gamma). \qquad (10)$$

First we take the n-segment of γ to be exactly n_α. For all terms $q_{\alpha(1)}, ..., q_{\alpha(n)}$, $|q_{\alpha(1)} - q_{\beta(1)}| < \frac{1}{2^0}$, ..., $|q_{\alpha(n)} - q_{\beta(n)}| < \frac{1}{2^{n-1}}$, by the definition of hypothesis $|\alpha - \beta| < \frac{1}{2^n}$. Next term γ_{n+1} has to satisfy both of the following inequalities

$$|q_{\gamma(n+1)} - q_{\gamma(n)}| = |q_{\gamma(n+1)} - q_{\alpha(n)}| < \frac{1}{2^{n+1}}, \qquad (11)$$

and

$$|q_{\gamma(n+1)} - q_{\beta(n+1)}| < \frac{1}{2^n}. \qquad (12)$$

[69]Note that θ is also an intuitionistic function in our sense, if Φ was given as an intuitionistic Function in our sense.

A simple line figure with $q_{\alpha(n)}$ and $q_{\beta(n)}$ and the corresponding intervals with them as centers and of length $\frac{1}{2^n}$ shows that there is always a rational $q = q_\lambda$ in the fixed enumeration satisfying the above inequalities. Therefore, we define $\gamma(n+1) = \lambda$. By that way we proceed at each step defining the element γ of (10).
By the definition of Φ though, $\Phi(\beta) = \Phi(\gamma)$. Therefore, the inequality $|\Phi(\alpha) - \Phi(\gamma)| < \frac{1}{2^m}$ becomes

$$|\Phi(\beta) - \Phi(\alpha)| < \frac{1}{2^m}. \qquad \diamond$$

In [Veldman 1999] we also find his treatment of NCT. Veldman claims to show NCT directly i.e., in a stronger way, rather than finding a weak counterexample. His proof though, uses CP as an axiom, while Brouwer's proof of BNCT is CP-free.

First we prove a proposition which is standard in post-Brouwer expositions of CP.

Proposition 8.3 (Negation of a form of the principle of the excluded middle NPEM) (CP is used as an axiom):

$$\neg[(\forall \alpha \in \omega^\omega)((\alpha = \overline{0}) \vee (\alpha \neq \overline{0}))].$$

Proof: Suppose that

$$(\forall PEM) \qquad (\forall \alpha \in \omega^\omega)((\alpha = \overline{0}) \vee (\alpha \neq \overline{0})).$$

Then, we may define on ω^ω the following function:

$$\varphi(\alpha) = \begin{cases} 1 & , \ \alpha \neq \overline{0} \\ 0 & , \ \alpha = \overline{0} \end{cases}$$

Hence, by CP, there is a natural number N such that each sequence β, N-same to $\overline{0}$, takes the value 0. But, as we have already said in Paragraph 6, this is absurd, since there is a sequence β, N-same to $\overline{0}$, which is not equal to $\overline{0}$, therefore it is mapped to 1 under φ.$\diamond$

Although PEM in the form $(P \vee \neg P)$ cannot be refuted, since in the intuitionistic propositional calculus

$$\neg\neg(P \vee \neg P)$$

is proved, the form $\forall PEM$ of PEM, or its obvious generalization

$$(\forall \alpha)(P(\alpha) \vee \neg P(\alpha)),$$

is standardly considered refuted. Actually, in the following proposition Veldman considers the following special case

$$(\forall \alpha \in \omega^\omega)((\alpha \neq \overline{0}) \vee \neg(\alpha \neq \overline{0}))$$

to be refuted.
From our point of view though, the above result is valid only in an axiomatic framework regarding intuitionistic analysis where the concept of intuitionistic function is only externally understood. Within our definition of function, the function φ of previous proposition is *not* an intuitionistic function at the first place, since there is no function φ^* given, which determines an initial segment of $\overline{0}$ responsible for the value of $\overline{0}$ under

φ. I.e., the above negation of $\forall PEM$ is based on an external concept of intuitionistic function, which is certainly outside Brouwer's spirit, and on a consequent axiomatic understanding of CP, which is fundamentally against Brouwer's conceptualism. In Brouwer's non-axiomatic spirit $\forall PEM$ is unacceptable through weak counterexamples and not strictly refuted. That would require a fundamentally different treatment of CP, from a self-evident or provable truth to an axiom.

Proposition 8.4 (Veldman's negative continuity theorem, (VNCT 1999): There is no real Function f such that:
(i) $f(0) = 1$, and
(ii) $f(\frac{1}{2^n}) = 0$, for each n.

Proof: Following [Veldman 2001], we suppose that there is such f and we finally reach a contradiction.
First we define the following sequences:
$e_0(n)$ is 1, if after the n-term of the decimal expansion of π there exist 98 consecutive 9's, while before that term this is not true, and $e_0(n)$ is 0 otherwise. Of course, until now we do not know if $e_0(n)$ is constantly 0 or not, since the existence of such a sequence of consecutive 9's is undecidable.
We define $t : \mathbb{N} \to \mathbb{N}$ s.t.,

$$q_{t(n)} = \frac{1}{2^n}, \quad \forall n.$$

Also, sequence β is defined by

$$\beta(n) = \begin{cases} t(n) & \text{, if } e_0(i) = 0,\ \forall i \leq n \\ t(i_0) & \text{, } i_0 \text{ min } \ i\text{: } e_0(i) \neq 0 \end{cases}$$

Thus, $\beta(n)$ equals $t(n)$ until the consecutive 9's are found and if they are found it becomes constant. If $e_0 = \overline{0}$, where $\overline{0}$ is the constant sequence of 0's, then $\beta(n) = t(n)$, for each n. Hence, $q_{\beta(n)} = q_{t(n)} = \frac{1}{2^n}$ and $\beta \approx \overline{0}$, since $|\frac{1}{2^n} - 0| = \frac{1}{2^n} < \frac{1}{2^{n-1}}$. Then, by the hypothesis on the existence of such an f, $f(0) = 1$, hence $f(\beta) = 1$.
If $e_0 \neq \overline{0}$ i.e., if the consecutive 9's were found, then

$$(\beta(n))_n = (t(1), t(2), ..., t(i_0 - 1), t(i_0), t(i_0), t(i_0), ...) = (t(1), t(2), ..., t(i_0 - 1), \overline{t(i_0)}),$$

and

$$(q_{\beta(n)})_n = (q_{t(1)}, q_{t(2)}, ..., q_{t(i_0-1)}, \overline{q_{t(i_0)}}) = (\frac{1}{2}, ..., \frac{1}{2^{i_0-1}}, \overline{\frac{1}{2^{i_0}}}).$$

Therefore, $\beta \approx \frac{1}{2^{i_0}}$, since $|q_{\beta(n)} - \frac{1}{2^{i_0}}| < \frac{1}{2^{n-1}}$, for each n, since the inequality holds trivially if $n \geq i_0$ and if $n < i_0$, clearly $|\frac{1}{2^n} - \frac{1}{2^{i_0}}| < \frac{1}{2^{n-1}}$. Since $\beta \approx \frac{1}{2^{i_0}}$, then, by the hypothesis on f, $f(\beta) = f(\frac{1}{2^{i_0}}) = 0$. In summary,

$$e_0 = \overline{0} \Rightarrow \beta \approx 0 \ \wedge \ f(\beta) = 1 \qquad (*)$$

$$e_0 \neq \overline{0} \Rightarrow \beta \approx \frac{1}{2^{i_0}} \ \wedge \ f(\beta) = 0. \qquad (**)$$

But we are unable to calculate $f(\beta)$, for we do not know how to define $(f(\beta))(2)$.
Consider $(f(\beta))(2)$ was known. Then $q_{(f(\beta))(2)}$ as a rational satisfies the following instance of the principle of the excluded middle:

$$(q_{(f(\beta))(2)} \geq \frac{1}{2}) \ \vee \ (q_{(f(\beta))(2)} < \frac{1}{2}). \qquad (\dagger)$$

Suppose first that $q_{(f(\beta))(2)} \geq \frac{1}{2}$. Then, $f(\beta) \neq 0$, since if $f(\beta) = 0$, then $|q_{(f(\beta))(2)} - 0| < \frac{1}{2}$, which is by hypothesis absurd. Therefore, $f(\beta) \neq 0$ and by $(**)$, we get $\neg\neg(e_0 = \overline{0})$. I.e.,

$$q_{(f(\beta))(2)} \geq \frac{1}{2} \Rightarrow \neg\neg(e_0 = \overline{0}).$$

If $q_{(f(\beta))(2)} < \frac{1}{2}$, then $f(\beta) \neq 1$, since if $f(\beta) = 1$, then $|q_{(f(\beta))(2)} - 1| < \frac{1}{2}$, which is absurd by our hypothesis on $q_{(f(\beta))(2)}$. Since $f(\beta) \neq 1$, then by $(*)$, $\neg(e_0 = \overline{0})$. I.e., we have proved that

$$q_{(f(\beta))(2)} < \frac{1}{2} \Rightarrow \neg(e_0 = \overline{0}).$$

So (†) led to

$$\neg\neg(e_0 = \overline{0}) \ \vee \ \neg(e_0 = \overline{0}),$$

which contradicts Proposition 8.3.$\diamond$.

Both proofs of NCT show that a certain real Function is not computable. In Brouwer's proof, if it was computable, it would mean that we would know the solution of a still unsolvable problem, while in Veldman's proof it would lead to an absurdity, through the negation of $\forall PEM$. Veldman claims that Brouwer proved NCT in a weak sense only, while if one wants to prove NCT in a strong sense, one needs CP. Surely, a proof of NCT in a strong sense needs CP to be used as an axiom, but this, in our opinion, is not a real win. To prove something strongly does not mean that we believe it more, since we have to explain the axioms used to provide its strong proof. We believe that if we want to preserve the definitional, non-axiomatic character of Brouwer's constructivism, we should not treat CP as an axiom, therefore we should not consider Proposition 9.3 as a real intuitionistic proposition. Although we do not know when this proposition appeared for the first time, we haven't found such a proposition in Brouwer's works. We tend to believe that Proposition 9.3 is a post-Brouwer proposition related to an axiomatic understanding of CP and an external conception of an intuitionistic function $\varphi : \omega^\omega \to \omega$.

9. The continuity principle for the intuitionistic Function $\Phi : \omega^\omega \to \omega^\omega$. We introduce the concept of an intuitionistic Function, defined on choice sequences of a spread and taking values also on the choice sequences of a spread, in the same way we introduced an intuitionistic function in Paragraph 6. We stress though, what we have already mentioned in previous paragraph, that Brouwer never gave an internal definition of an intuitionistic Function $\Phi : \omega^\omega \to \omega^\omega$, although he implied an internal concept of an intuitionistic function $\varphi : \omega^\omega \to \omega$. Our reconstruction of intuitionistic mappings and the consequent treatment of the corresponding continuity principles as theorems derived from definitions rather than axioms is a deviation from Brouwer's writings but, in our view, not from Brouwer's spirit. We consider the following definition necessary, in order the concept of intuitionistic Function is understood constructively, a normative feature of all intuitionistic objects.

An *intuitionistic ω^ω-Function*, $\Phi : \omega^\omega \to \omega^\omega$, is a law which corresponds an ω^ω-sequence α to a unique ω^ω-sequence β, based on a law Φ^*, which correlates finite sequences of naturals such that:
(i) if $N \leq M$, then $\Phi^*(a_1, a_2, ..., a_N) \preceq \Phi^*(a_1, a_2, ..., a_M)$, where $\preceq$ means that the sequence $\Phi^*(a_1, a_2, ..., a_N)$ is an initial segment of the sequence $\Phi^*(a_1, a_2, ..., a_M)$. Note

that $\Phi^*(a_1, a_2, ..., a_N)$ may be the root <>.
(ii) Φ^* is not finally constant.
(iii) $\Phi(\alpha) = \sup_N \Phi^*(N_\alpha)$ i.e., $\Phi(\alpha)$ is approximated by the segments $\Phi^*(N_\alpha)$. Then, we say that Φ^* *computes* Φ.

This definition is natural, since the image of an on-going object through Φ is another on-going object. Of course, if $\Phi(\alpha) = \beta$, not every segment of β is the image of a segment of α under Φ^*. If M_1, M_2 are arbitrary spreads an (M_1, M_2)-*Function* $\Phi : M_1 \to M_2$ is defined in the same way.
As we show in the Appendix, the above definition is the continuity condition of a function $\Phi : \mathcal{N} \to \mathcal{N}$. Hence, the definition of an intuitionistic function $\Phi : \omega^\omega \to \omega^\omega$ is such that the following is automatically satisfied:

An intuitionistic Function $\Phi : \omega^\omega \to \omega^\omega$, interpreted classically, is always a continuous function.

We may give an intuitionistic meaning to this fact.
An intuitionistic Function $\Phi : \omega^\omega \to \omega^\omega$ is called *continuous* iff for each sequence α the following condition is satisfied:

$$\forall \lambda \in \omega \quad \exists k \in \omega, \text{ such that, } \quad \beta, \alpha \ k\text{-equal} \Rightarrow \Phi(\beta), \Phi(\alpha) \ \lambda\text{-equal.}$$

Proposition 9.1 (Continuity Principle of Intuitionistic Functions (CPF)): An intuitionistic Function $\Phi : \omega^\omega \to \omega^\omega$ is always continuous.

Proof: Let α be any sequence of the universal spread ω^ω. The λ-initial segment of $\Phi(\alpha)$ is by hypothesis determined by some k-initial segment of α. The natural number k can be found effectively as follows: We calculate finite sequences $\Phi^*(\alpha_1), ..., \Phi^*(\alpha_1, \alpha_2, ..., \alpha_k)$ until we reach or surpass the λ-initial segment of α for the first time.
Obviously, each k-same to α sequence β is such that $\Phi(\beta)$ is λ-same to $\Phi(\alpha)$.$\diamond$

The concept of a continuous Function is not a replica of the classical one, but it has an intuitionistic meaning. The continuous property reflects the fact that the calculation of any λ-segment of a sequence $\Phi(\alpha)$ doesn't only expresses the λ-knowledge of $\Phi(\alpha)$, but also the λ-knowledge of any k-same sequence to α. Therefore, through the gradual determination of $\Phi(\alpha)$ the values of a species of sequences is gradually determined.
Proposition 9.1 can be considered as a continuity principle (CPF) for an intuitionistic Function, and as in the function case, it is a direct result of its definition. The above definition though, is the necessary result of the action of Φ on on-going objects with values also on on-going objects.

(I) Φ corresponds to the on-going object α an on-going object β. This is done necessarily through some Φ^*, since we only know initial segments of the on-going objects α.
(II) As the input information i.e., the length of the initial segments of α, grows, our knowledge of the output sequence i.e., the length of the initial segments of the output sequence β, has to grow too. Of course, Φ^* must not be finally constant, if we want to find an infinite sequence β as the value of α under Φ. Condition (i) of our definition is necessary if we want a gradual knowledge of $\Phi(\alpha)$. If $\Phi^*(a_1, a_2, ..., a_N), \Phi^*(a_1, a_2, ..., a_M)$ were not related, then we wouldn't have any partial knowledge of $\Phi(\alpha)$ at any stage of the formation of α.
(III) The value of α under Φ, because of condition (ii) of our definition, cannot be other than the on-going object

$$\Phi^*(a_1)$$
$$\Phi^*(a_1, a_2)$$
$$\dots\dots\dots$$
$$\Phi^*(a_1, a_2, ..., a_N)$$
$$\dots\dots\dots\dots\dots$$

Suppose someone defined intuitionistic Function $\Phi : \omega^\omega \to \omega^\omega$ as follows:

If $\varphi_1, \varphi_2, ..., \varphi_n, ...$ is a constructively given sequence of intuitionistic functions, then let $f : \omega^\omega \to \omega^\omega$ be the object satisfying

$$f(\alpha) = (\varphi_1(\alpha), \varphi_2(\alpha), ..., \varphi_n(\alpha), ...).$$

At first sight, this seems to be an equally good way to define an intuitionistic Function. Moreover, if $\varphi_n(\alpha) = \alpha(n)$ the identity Function is determined, while if $\varphi_n(\alpha) = n$ a constant Function is determined corresponding to each sequence of ω^ω the sequence of ω.

Apart from the fact that the concept of a constructively sequence is not specified, the following problem is found with regard to the above definition :

The values of $\varphi_n(\alpha)$ are not independent from us, but they depend on some initial segments of α. Since α is gradually generated, at each stage of its formation only some sequences $\varphi_n(\alpha)$ are generated. Meaning that at each moment we do not know any initial segment of $f(\alpha)$, since it is possible that it is not enough to generate e.g., φ_1^*. It is though, essential to our knowledge of an on-going object that at each moment we posses a partial knowledge of it. But,

(i) *Partial knowledge of an on-going object intuitionistically means knowledge of an initial part of it.*

(ii) *Knowledge of an on-going object intuitionistically means the gradual and ever-growing partial knowledge of it.*

Only our initial definition is compatible with (i) and (ii), being in "parallel" to the definition of a spread.

If M_1, M_2 are spreads, then an *intuitionistic* (M_1, M_2)*-Function* $\Phi : M_1 \to M_2$ is defined similarly and a (M_1, M_2)-Function is proven continuous likewise.

A Function $\Phi : M_1 \xrightarrow{1-1} M_2$ and onto M_2 is a *homeomorphism* iff there is a Function $\Phi^{-1} : M_2 \to M_1$, such that $\Phi \circ \Phi^{-1} = id_{M_2}$ and $\Phi^{-1} \circ \Phi = id_{M_1}$.

CPF is classically false: As we show in the Appendix, there exist non-continuous functions $f : \mathcal{N} \to \mathcal{N}$. We consider there the following map:

$$f(\alpha) = \begin{cases} \overline{0} & \text{, if } \alpha = \overline{0} \\ \overline{1} & \text{, if } \alpha \neq \overline{0} \end{cases}$$

This map though, is not an intuitionistic Function, since there is no monotone f^* that computes f. If there was such f^*, then, for each n there is m such that,

$$f^*(\underbrace{0, 0, ..., 0}_{n}) = (\underbrace{0, 0, ..., 0}_{m}).$$

Since $f^*(\underbrace{0, 0, ..., 0}_{n}, 1)$ is an initial segment of $\overline{1}$, then in both cases,

$$f^*(\underbrace{0, 0, ..., 0}_{n}, 1) = <>$$

or

$$f^*(\underbrace{0,0,...,0}_{n},1) = (\underbrace{1,1,...,1}_{k}),$$

for some k, monotonicity is clearly violated for $(\underbrace{0,0,...,0}_{n}) \prec (\underbrace{0,0,...,0}_{n},1)$.

It is easily seen though, that the standard injection $\Theta : \mathcal{N} \to \mathcal{C}$ is intuitionistically accepted (see the Appendix for the definition of Θ).

Proposition 9.2: If M is a non-empty spread, there is a retraction $\Theta : \omega^\omega \to M$ i.e., an intuitionistic (ω^ω, M)-Function which is the identity on M[70].

Proof: It suffices to define a function $\Theta^* : \omega^{<\omega} \to M^{<\omega}$, which is monotone, not finally constant and calculates Θ.
Let (a_1) any 1-sequence of natural numbers. We then define:

$$\begin{aligned} (a_1) \overset{\Theta^*}{\mapsto} & (a_1), \text{ if } (a_1) \text{ is } \Lambda_M\text{-accepted.} \\ & (b_1), \text{ if } (a_1) \text{ is not } \Lambda_M\text{-accepted and } (b_1) \hookrightarrow (a_1), \end{aligned}$$

where $(b_1) \hookrightarrow (a_1)$ means that (b_1) is the most close to (a_1) M-sequence in the following sense: number b_1 is the largest number of those numbers smaller than a_1 such that (b_1) is M-accepted, or it is the smallest number of those larger than a_1 such that (b_1) is M-accepted, if none of the numbers smaller than a_1 does not form a Λ_M-accepted 1-sequence. Since Λ_M is decidable and M is non-empty, the above procedure terminates in finite time.
Let (a_1, a_2) any 2-sequence of natural numbers. We then define:

$$\begin{aligned} (a_1,a_2) \overset{\Theta^*}{\mapsto} & (a_1,a_2), \text{ if } (a_1,a_2) \text{ is } \Lambda_M\text{-accepted} \\ & (a_1,b_2), \text{ if } (a_1) \text{ is } \Lambda_M\text{-accepted, } (a_1,a_2) \text{ is not } \Lambda_M\text{-accepted} \\ & \quad \text{and } (b_2) \hookrightarrow (a_2) \\ & (b_1,b_2), \text{ if } (a_1) \text{ is not } \Lambda_M\text{-accepted and } (b_1,b_2) \hookrightarrow (a_1,a_2), \end{aligned}$$

where the expression $(b_2) \hookrightarrow (a_2)$ is interpreted as before for the Λ_M-accepted sequence (a_1, b_2). Also, expression $(b_1, b_2) \hookrightarrow (a_1, a_2)$ is interpreted through $(b_1) \hookrightarrow (a_1)$ and $(b_2) \hookrightarrow (a_2)$ for the Λ_M-accepted sequence (b_1, b_2).
Defining Θ^* analogously on any sequence $(a_1, a_2, ..., a_n)$ of natural numbers, we get the desired Θ.$\diamond$

Retraction $\Theta : \omega^\omega \to M$ transfers properties of the universal spread to an arbitrary spread M. As we have already said in Paragraph 6

$$CP(M) \qquad \forall\alpha(\alpha \in [M])(\exists N)(\forall\beta, N_\beta = N_\alpha \Rightarrow \varphi(\beta) = \varphi(\alpha))$$

is the continuity principle for an arbitrary spread M. Next proposition shows that CP(M) is a consequence of continuity principle CP.

Proposition 9.3: CP $\Rightarrow$ CP(M).

Proof: Each function φ_M defines the universal function $\varphi = \varphi_M \circ \Theta$. Hence, CP applied on φ,

$$\forall\alpha(\alpha \in \omega^\omega)(\exists N)(\forall\beta, N_\beta = N_\alpha \Rightarrow \varphi_M \circ \Theta(\beta) = \varphi_M \circ \Theta(\alpha)),$$

[70] Proposition 9.2 is trivially true in case M is the empty spread.

gives directly CP(M).$\diamond$

Intuitionistic functions can be seen as a special kind of intuitionistic Functions. Let Ω be the spread of all constant sequences $\overline{n} = (n, n, n, ...)$. Ω is a spread and not a fan, since the root $<>$ has infinitely many immediate successors. Then, we prove the following proposition.

Proposition 9.4: (i) If $\varphi : \omega^\omega \to \omega$ is an intuitionistic function, there is an intuitionistic Function $F_\varphi : \omega^\omega \to \Omega$, such that $F_\varphi(\alpha) = \overline{\varphi(\alpha)}$.
(ii) If $F : \omega^\omega \to \Omega$ is an intuitionistic Function, there is an intuitionistic function $\varphi_F : \omega^\omega \to \omega$, such that $\varphi_F(\alpha) = n$, if $F(\alpha) = \overline{n}$.

Proof: (i) Let α be a sequence in ω^ω. By CP on φ, there is some N, such that $N_\alpha = N_\beta \Rightarrow \varphi(\alpha) = \varphi(\beta) = n$, for each β in ω^ω. We then define F^*_φ by

$$F^*_\varphi(m_\alpha) = \begin{cases} <> & \text{, if } m < N \\ (\underbrace{n, n, ..., n}_{m}) & \text{, if } m \geq N \end{cases}$$

Obviously, F^*_φ is monotone, non-stagnant and computes Function F_φ, satisfying $F_\varphi(\alpha) = \overline{\varphi(\alpha)}$.
(ii) If α is again a fixed sequence in ω^ω, let N be (for example) the first natural such that $F^*(N_\alpha)$ is a sequence other than the root. Let $F^*(N_\alpha) = (\underbrace{n, n, ..., n}_{m})$, for some natural m. We then define $\varphi_F(\alpha) = n$. Obviously, if β is any other sequence such that $N_\beta = N_\alpha$, then $F^*(N_\beta) = (\underbrace{n, n, ..., n}_{m})$, and since β belongs to Ω, $F(\beta) = F(\alpha) = \overline{n}$ and $\varphi_F(\beta) = \varphi_F(\alpha) = n$.$\diamond$

Obviously,

$$\varphi = \varphi_{F_\varphi},$$

while Proposition 10.4(ii) shows that

$$CPF \Rightarrow CP.$$

Hence, it is no surprise that intuitionistic Function is instrumental to the proof of negative continuity theorem of an intuitionistic function.
An *external* function $\varphi : \omega^\omega \to \omega$ is a law which corresponds sequences to naturals without any explication of how this is done. Of course, a classical function is an external kind of function. Our concept of intuitionistic function φ is an *internal* function, since φ, not only sends sequences to naturals, but also it is inherent to φ the way this correspondence is established.
In [van Atten, van Dalen 2002], pp.341-2, we find the following proposition, the proof of which is given without using CP:

Proposition 9.5 (Negative continuity theorem - without CP): There is no non-continuous external function $\varphi : \omega^\omega \to \omega$.

Proof: We suppose that φ is non-continuous, without loss of generality on the constant zero function $\overline{0}$, and $\varphi(\overline{0}) = 0$. Thus, in order to extract a weak counterexample, we have supposed that:

(i) $(\forall N)(\exists\alpha)(N_\alpha = N_0 \wedge \varphi(\alpha) \neq \varphi(\overline{0})$.
(ii) $\varphi(\overline{0}) = 0$.

(i) results from the negation of continuity at $\overline{0}$ and expresses the fact each input $N_{\overline{0}}$ of $\overline{0}$, cannot activate $\varphi(\overline{0})$, since if it could, (i) would be violated. Therefore, our hypothesis is equivalent to the following:

(I) None $N_{\overline{0}}$ activates $\varphi(\overline{0})$.
(II) At the same moment $\varphi(\overline{0}) = 0$.

Hence, φ is not an intuitionistic function, but an externally defined linguistic mapping concept. For this external φ though, van Atten and van Dalen extract the following weak counterexample:
Let α_N the selected by (i) sequence a corresponding to N. Each α_N has its N-segment on the infinite branch $\overline{0}$ and at some successor node it branches off $\overline{0}$. We may assume that α_{N+1} branches off later than α_N. The following spread S is defined by

$$u \in S \Leftrightarrow (\exists N)\ (u \in \alpha_N),$$

and the Function $F : \omega^\omega \to [S]$, computed by $F^* : \omega^\omega \to S$, where

$$u \mapsto r_u,$$

and r_u is the rightmost node of S to the left of u with the same length.
Obviously, $u \preceq v \Rightarrow r_u \preceq r_v$, since a successor of r_u is closer to v, and F^* is non-stagnant. Hence, $F(\alpha) = lim_n F^*(n_\alpha)$. If we define

$$\alpha \sharp \beta \Leftrightarrow \exists i,\ \alpha(i) \neq \beta(i)$$

i.e., a strong, positive kind of inequality of sequences ($\alpha\sharp\beta \Rightarrow \alpha \neq \beta$, but not the inverse), then:
$(*)$ $(\forall\alpha)(\alpha\sharp\overline{0} \Rightarrow \varphi(F(\alpha)) \neq 0)$.
$(**)$ $(\forall\alpha)(\varphi(F(\alpha)) \neq 0 \Rightarrow \neg\neg\alpha\sharp\overline{0})$.
$(*)$: If $\alpha\sharp\overline{0}$, then α branches off $\overline{0}$ at some node u, $F(\alpha) \in [S] - \{\overline{0}\}$ and the value $\varphi(F(\alpha)) \neq 0$.
$(**)$: Since $\neg(\exists i,\ \alpha(i) \neq 0) \Leftrightarrow \forall i,\ \alpha(i) = 0$ i.e.,

$$\neg\alpha\sharp\overline{0} \Leftrightarrow \alpha = \overline{0},$$

then $F(\alpha) = F(\overline{0}) = \overline{0}$, by the definition of F ($\overline{0} \in [S]$). Hence, $\varphi(F(\alpha)) = \varphi(F(\overline{0})) = \varphi(\overline{0}) = 0$, which is by hypothesis absurd. Hence, $\neg\neg\alpha\sharp\overline{0}$ is proven. Since within BHK-interpretation of quantifiers[71]

$$(P \Rightarrow Q) \Rightarrow (\neg Q \Rightarrow \neg P),$$

holds, but not the inverse, then applying the above scheme to $(*)$ and $(**)$, we get $\varphi(F(\alpha)) = 0 \Rightarrow \alpha = \overline{0}$ and $\neg\neg\neg\alpha\sharp\overline{0} \Rightarrow \varphi(F(\alpha)) = 0$ respectively. Since within BHK though,

$$\neg\neg\neg P \Leftrightarrow \neg P$$

[71] BHK stands for Brouwer, Heyting and Kolmogorov.

also holds, we get $\alpha = \overline{0} \Rightarrow \varphi(F(\alpha)) = 0$. Thus, finally we get

$$(\forall \alpha)(\alpha = \overline{0} \Leftrightarrow \varphi(F(\alpha)) = 0).$$

But then,

$$(\dagger) \quad (\forall \alpha)(\alpha = \overline{0} \ \vee \ \alpha \neq \overline{0}),$$

since for each a, $\varphi(F(\alpha))$ is effectively computed, and either $\varphi(F(\alpha)) = 0$ i.e., $\alpha = \overline{0}$, or $\varphi(F(\alpha)) \neq 0$ i.e., $\alpha \neq \overline{0}$. (†) though, known as the weak limited of omniscience (WLPO) or $\forall PEM$, is an intuitionistically unaccepted formula, by a standard Brouwerian counterexample.$\diamond$

The above impossibility result owes its existence to Brouwer's negative continuity theorem (Proposition 8.1) of real functions defined on the unit continuum. As we have discussed in Paragraph 8, Brouwer had an internal concept of intuitionistic function in his mind, while he worked with an external concept of a real Function i.e., a mapping on point cores and values point cores. In Paragraph 8 we explained why this asymmetry between intuitionistic functions and real Functions is problematic. In the case of a function defined on sequences i.e., on on-going objects, or *objects in time*, the use of an external, timeless concept of function is completely against Brouwer's ideas. Of course, Brouwer himself is not consistent to an internal concept of function when the values are sequences too. In our opinion though, negative continuity theorems seem not that important to us, since they refer to a somehow classical concept of function. An external φ functions in a magical linguistic way, exactly like a classical function. We believe that a mapping on on-going objects should be influenced in its structure from the on-going character of the objects on which it is applied. Also, if we want to create a constructive theory of the continuum, then each mathematical object involved must correspond to some construction. *An externally defined function lacks constructive content.* This is the reason why we defined intuitionistic Functions $\Phi : \omega^\omega \to \omega^\omega$ internally, in complete analogy to intuitionistic functions $\varphi : \omega^\omega \to \omega$.

10. Well-ordered species and bars. Brouwer's Fan theorem (BFT) is in the core of intuitionistic analysis. Through BFT Brouwer managed to prove his (highly non-classical) Uniform Continuity theorem (UCT). The proof of BFT not only determined the character of Brouwer's intuitionistic analysis (BIA), but also its post-Brouwer development.

Brouwer's first proof of BFT, in [Brouwer 1924a][72], appears two years before the proof of König's lemma, a proposition which is classically equivalent to BFT. As we have already seen, König's lemma is highly non-constructive, since it guarantees the logical only existence of an infinite branch in an infinite fan. A genuine construction of the infinite branch seems impossible in a classical framework. Historically, the two theorems are not related, though Fan theorem is usually mentioned as an outstanding example of a proposition the intuitionistic proof of which preceded its classical proof.

The proof of BFT that we present here is in [Brouwer 1927][73]. BFT is the following proposition:

[72]First suggested in [Brouwer 1923c] inconclusively.

[73]This is the most "standard" one, the other two proofs are found in [Brouwer 1924a], as we have already mentioned, and in [Brouwer 1954]. Heyting's proof of BFT found in [Heyting 1956] is based on [Brouwer 1924a] proof.

Brouwer's Fan theorem: If T is a fan and $\varphi : T \to \omega$ an intuitionistic function, then there exists a natural N, such that, for each infinite T-sequence α, its value $\varphi(\alpha)$ is determined by its initial segment N_α. I.e.,

$$(*) \qquad \exists N \forall \alpha (\alpha \in [T])(\varphi(\alpha) = \varphi^*(N_\alpha))$$

By Continuity Principle on a fan T, which is a (finitely branching) spread, we get

$$(**) \qquad \forall \alpha (\alpha \in [T])(\exists N)(\forall \beta, N_\beta = N_\alpha \Rightarrow \varphi(\beta) = \varphi(\alpha))$$

The interchange of quantifiers in $(*)$ and $(**)$ is the same to the interchange of quantifiers in the definitions of uniform continuity (1) and continuity (2) of a classical function $f : X \to \mathbb{R}$, where $X \subseteq \mathbb{R}$ and $\mathbb{R}$ is the classical set of real numbers. I.e., for each $\varepsilon > 0$:

$$(1) \qquad (\exists \delta)(\forall x)(\forall y)(|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon,$$

$$(2) \qquad (\forall x)(\exists \delta)(\forall y)(|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

This fact is not accidental, since, as we said at the beginning, BFT is a central tool in Brouwer's proof of UCT.
While CP(T) for a fan T expresses the fact that an initial segment N_α of a choice sequence α determines the value $\varphi(\alpha)$, where $\varphi : T \to \omega$ is an intuitionistic function, and that N is constructively given by the way φ is defined, BFT expresses a "global" version of CP. According to BFT, a natural number N can constructively be found such that the N-initial segment of *each* choice T-sequence α determines its value $\varphi(\alpha)$.

BFT taken verbatim does not hold classically.

Consider, for example, the function $\varphi : \mathcal{C} \to \omega$, computed by the following $\varphi^* : 2^{<\omega} \to \omega$:

$$\begin{gathered}
\varphi^*(1) = 1 \\
\varphi^*(0,1) = 2 \\
\cdots\cdots\cdots\cdots \\
\varphi^*(\underbrace{0, 0, ..., 0}_{n}, 1) = n + 1 \\
\cdots\cdots\cdots\cdots \\
\varphi(\overline{0}) = 0
\end{gathered}$$

$\mathcal{C}$ is a fan and φ is a classically accepted function on $\mathcal{C}$ which cannot be determined by a global bound N. Although φ is "algorithmically" defined on Cantor space $\mathcal{C}$, it is not $\mathcal{C}$-continuous at the zero sequence $\overline{0}$, since the non-zero sequences that extend the initial segment $\underbrace{0, 0, ..., 0}_{n}$ have $\mathcal{C}$-distance $< \frac{1}{n}$, while their values under φ have distance ≥ 1. But φ is *not* an intuitionistic, since its value on $\overline{0}$ is not determined by any of its initial segments. The essential difference between 2^ω and $\mathcal{C}$ is that $\mathcal{C}$-sequences, like $\overline{0}$, have an existence independent from their generation, while 2^ω-sequences exist only as Λ_{2^ω}-procedures.

BFT though, holds classically if it is understood as follows: **Each continuous function on a fan is uniformly continuous**. The above function φ violates BFT but also CP i.e., it is not even continuous. If we restrict to the fan $\mathcal{C}$, BFT($\mathcal{C}$) expresses that each continuous function on $\mathcal{C}$ is uniformly continuous. If $f : \mathcal{C} \to \mathbb{N}$, the uniform continuity property of them becomes

$$(\exists N)(\forall \alpha, \beta) \;\; N_\alpha = N_\beta \Rightarrow f(\alpha) = f(\beta),$$

as we show in the Appendix. Thus, BFT directly expresses the uniform continuity of an $f : \mathcal{C} \to \mathbb{N}$, if we interpret the binary fan and the species of naturals classically.

Usually, as for example in [Kleene, Vesley 1965], the CP is treated as the only purely intuitionistic principle. That is why CP is presented in [Kleene, Vesley 1965] last, after Bar induction, which is in Kleene's system an axiom necessary for the proof of fan theorem, and classically valid. Actually, Kleene, in [Kleene 1969], defined a formal system which gives classical analysis, if the PEM is added to it, and also Kleene's intuitionistic analysis (FIM), if a form of CP is added to it.

Our approach differs crucially from that of Kleene. Firstly, according to our need to reconstruct Brouwer's intuitionism and reveal its relevance to a general constructive spirit, we present Brouwer's original non-classical theorem. Secondly, while CP is an immediate consequence of the intuitionistic function concept, BFT is a highly non-trivial intuitionistic truth, which begs a genuine intuitionistic proof, by the analysis of the concepts involved in its formulation, as in the proof of CP, and not a simple derivation from an axiom like bar induction. Also, it should follow CP since it presupposes the concept of an intuitionistic function.

Before giving Brouwer's proof we need to define some new concepts and prove some simple results on them.

Classically, a well-ordered set Ω is an ordered set such that each non-empty subset of Ω has a first element. The following weak counterexample shows why this concept fails from the standard intuitionistic point of view.

Let $\Omega = \{0, 1\}$, where $0 < 1$ and $A = \{n = 0 \mid P \vee \neg P\}$, where P is open i.e., it is not known neither if P nor $\neg P$ is true. Obviously, A is a subspecies of $\{0, 1\}$. If A is the empty species, then we get $\neg P \vee \neg P$, which is absurd, since within BHK $\neg\neg(P \vee \neg P)$ can be proven. If A is $\{0\}$, then we get $P \vee \neg P$, which intuitionistically means that P is known to be true, or that $\neg P$ is known to be true, contradicting our hypothesis on P. Therefore, we cannot determine the first element of A, which is necessary to assert, intuitionistically, that A has a first element.

Brouwer, following some original ideas of Cantor on well-ordering, defined in his dissertation the well-ordered sets as the sets generated by singletons and then new well-ordered sets are constructed by "putting together" finite or potentially infinite sequences of already constructed well-ordered sets. Their conceptual justification was based on the at most ω-repetition of the same operation.

Before we give the inductive definition of Brouwer's well-ordered species, a concept of Brouwer's mature period which replaced that of well-ordered set from his early period, we need to define the union of species.

The (intuitionistic) *union* $E \vee Z$ of already constructed species E, Z is the species of objects a satisfying the disjunction

$$(\alpha \in E) \vee (\alpha \in Z)$$

i.e., an element a satisfies (or "belongs to") the species $E \vee Z$ iff we know that one of the two terms of the disjunction is satisfied by a.

In a similar way, the intuitionistic (potentially) *infinite union* $\bigvee_{i=1}^{\infty} E_i$ of a sequence of already constructed species $E_1, E_2, ..., E_n, ...$, given, as a sequence, in an intuitionistically accepted way, is defined as the species of objects a satisfying

$$\alpha \in \bigvee_{i=1}^{\infty} E_i \leftrightarrow (\alpha \in E_1) \vee ... \vee (\alpha \in E_n) \vee ...$$

i.e., an element a satisfies (or "belongs to") the species $\bigvee_{i=1}^{\infty} E_i$ iff we know that one of the infinite terms of the disjunction is satisfied by a.

The intuitionistic *conjunction* $E \wedge Z$ of already constructed species E, Z is the species of objects a satisfying the conjunction

$$(\alpha \in E) \wedge (\alpha \in Z)$$

i.e., an element a satisfies (or "belongs to") the species $E \wedge Z$ iff we know that both of the two terms of the conjunction are satisfied by a.
In a similar way the intuitionistic (potentially) *infinite conjunction* $\bigwedge_{i=1}^{\infty} E_i$ of a sequence of already constructed species $E_1, E_2, ..., E_n, ...$, given, as a sequence, in an intuitionistically accepted way, is defined as the species of objects a satisfying

$$\alpha \in \bigwedge_{i=1}^{\infty} E_i \leftrightarrow (\alpha \in E_1) \wedge ... \wedge (\alpha \in E_n) \wedge ...$$

i.e., an element a satisfies (or "belongs to") the species $\bigwedge_{i=1}^{\infty} E_i$ iff we know that each of the infinite terms of the conjunction is satisfied by a.

A *well-ordered species* (w.o.s) A is defined inductively as follows:
(i) if A is one-element species, then it is a w.o.s. The element of an one-element species is a natural number or an element of some *decidable species* i.e., a species for which there is an effective answer to the question whether $\alpha \in A$ or $\alpha \notin A$. Here we use an informal concept of an effective procedure.
(ii) if $A_1, A_2, ..., A_n$ are disjoint[74] w.o.s, then their ordered sum $\bigoplus_{i=1}^{n} A_i$ is a w.o.s, where $\bigoplus_{i=1}^{n} A_i$ is the union of A_i such that, each A_i preserves its order and $j < k$, then $a \prec b$, for $a \in A_j$ and $b \in A_k$.
(iii) if $A_1, A_2, ..., A_n, ...$, is a constructively given (i.e., given through an algorithm[75]) sequence of disjoint w.o.s, then their infinite ordered sum $\bigoplus_{i=1}^{\infty} A_i$ is a w.o.s, where $\bigoplus_{i=1}^{\infty} A_i$ is defined analogously.

Hence, a non-trivial w.o.s is a structure of the form

$$\bigoplus_{i=1}^{n} A_i = (\bigvee_{i=1}^{n} A_i, \prec),$$

or of the form

$$\bigoplus_{i=1}^{\infty} A_i = (\bigvee_{i=1}^{\infty} A_i, \prec).$$

[74] Two species E, Z are called *disjoint* if we have shown the impossibility for an object a to belong to both E and Z.

[75] Hence, the sequence of A_i's is not an on-going object.

By the definition of union, in the case of the sum $\bigoplus_{i=1}^{n} A_i$,

$$\alpha \in \bigoplus_{i=1}^{n} A_i \leftrightarrow (\alpha \in A_1) \vee ... \vee (\alpha \in A_n)$$

i.e., $\alpha \in \bigoplus_{i=1}^{n} A_i$ iff we know in which A_i α belongs to.
Also, in the case of the infinite sum $\bigoplus_{i=1}^{\infty} A_i$ we mean that

$$\alpha \in \bigoplus_{i=1}^{\infty} A_i \leftrightarrow (\alpha \in A_1) \vee ... \vee (\alpha \in A_n) \vee ...$$

i.e., $\alpha \in \bigoplus_{i=1}^{\infty} A_i$ iff we know in which A_i α belongs to.
Note that, as in any definition of a Brouwerian concept, the definition of a w.o.s refers to a certain construction. A well-ordered species corresponds to a certain executed construction of the mind.
A *w.o.s of first kind* is a w.o.s formed by one-element species and finite only ordered sums, while a *w.o.s of second kind* is a w.o.s formed by infinite ordered sums too. E.g., ω itself is a w.o.s with its natural order and it can be seen as the infinite sum $\bigoplus_{i=1}^{\infty}\{i\}$, where $\{i\}$ is the one-element w.o.s of the natural number i. In analogy to the definition of countable ordinals we get the following types of w.o.s of the second kind.
If A is a w.o.s of the form $\bigoplus_{i=1}^{\infty} A_i$, such that each A_i is an one-element species, it is called a w.o.s of *ω-type*. A w.o.s Ω is of $\omega+1$-type iff $\Omega = A \bigoplus \{a\}$, where $A = \bigoplus_{i=1}^{\infty}\{a_i\}$ is a w.o.s of ω-type and $a \neq a_i$, for each i. Similarly we define w.o.s of $\omega+2$, $\omega+3$-type etc., and $\omega+\omega$ or $\omega 2$-type, if $\Omega = A \bigoplus B$, where A, B are of ω-type and the appropriate disjointness condition is satisfied. Similarly w.o.s of $\omega 3$-type are defined and going on like that of *ω^2-type* iff $\Omega = \bigoplus_{i=1}^{\infty} A_i$, where A_i is a w.o.s of ω-type and the appropriate disjointness condition is satisfied. Going on like that, we can define w.o.s of $\omega^2+\omega$, $\omega^2+\omega 2$, ..., $\omega^2+\omega^2 = \omega^2 2$-type and then of $\omega^3, \omega^4, ..., \omega^\omega, ..., \omega^{\omega^\omega}, ..., \varepsilon_0$-type.
Let α be a choice sequence of a spread M such that $\alpha_i \neq \alpha_j$. The terms of α do not follow necessarily a preexisted law of generation, since M can be a subjective spread. Let $A_i = \{\alpha_i\}$. Then, we cannot talk of the w.o.s $A = \bigoplus_{i=1}^{\infty} A_i = \bigoplus_{i=1}^{\infty}\{\alpha_i\}$, since the sequence of A_i's is not given through an algorithm.

To the above inductive definition corresponds naturally the following inductive scheme.

Proposition 10.1 (induction scheme of w.o.s, IWOS): Let $\mathcal{P}$ a constructively accepted property on well-ordered species A, satisfying the following conditions:
(i) If A is an one-element w.o.s, then $\mathcal{P}(A)$ i.e., A satisfies property $\mathcal{P}$.
(ii) If $A = \bigoplus_{i=1}^{n} A_i$ and for each i from 1 to n, $\mathcal{P}(A_i)$, then $\mathcal{P}(A)$.
(iii) If $A = \bigoplus_{i=1}^{\infty} A_i$ and for each i, $\mathcal{P}(A_i)$, then $\mathcal{P}(A)$.
Then, for each w.o.s A, A satisfies $\mathcal{P}$ i.e., intuitionistically understood

$$(\forall A)\mathcal{P}(A).$$

Proof: Let A be a given w.o.s. That means that we have enough information in order to fully describe the *ordinal tree* $\mathcal{T}_A$ corresponding to A. Each branch of $\mathcal{T}_A$ is finite (although $\mathcal{T}_A$ is not, in general, a fan) and its end nodes are one-element species on which P holds. Going upwards from the end nodes to A, we do so through finite or infinite direct sums and consequently P is transferred from the one-element species up to A.

The above procedure is effective, since the sequences of w.o.s are given effectively.$\diamond$

We can now prove inductively the following propositions.

Proposition 10.2: If A is a w.o.s such that $\alpha \in A$, then there is a w.o.s B constituent in the structure of A i.e., a building block sub-species of A, such that $\alpha \in B$.

Proof: If A is an one-element species, then $B = A$. If $A = \bigoplus_{i=1}^{n} A_i$, or if $A = \bigoplus_{i=1}^{\infty} A_i$, then by the definition of finite or infinite ordered sums and the intuitionistic interpretation of finite or infinite disjunctions, there is some i such that $\alpha \in A_i$, therefore $B = A_i$. By IWOS, the conclusion runs through all w.o.s A.

Proposition 10.3: If A is a w.o.s, then A has a first element.

Proof: If A is an one-element species, then the first element exists in a trivial manner. If $A = \bigoplus_{i=1}^{n} A_i$, such that each A_i has a first element, then the first element of A is the first element of A_1. Similarly, in case $A = \bigoplus_{i=1}^{\infty} A_i$.$\diamond$

As we have shown at the beginning of this paragraph, we cannot say that any subspecies of a w.o.s A has a first element i.e., a w.o.s doesn't satisfy the classical definition of a well-ordered set. A itself though, always has one.

Proposition 10.4: If A is a w.o.s and $\alpha \in A$, then α has an immediate descendant or α is a last element i.e., there is no element of A larger than α.

Proof: If $A = \{\alpha\}$, then α is a last element. If $A = \bigoplus_{i=1}^{n} A_i$ and each A_i satisfies the inductive hypothesis, then $\alpha \in A$ means that $\alpha \in A_i$, for some i, therefore, the proposition holds. Similarly, if $A = \bigoplus_{i=1}^{\infty} A_i$.$\diamond$

Proposition 10.5: (i) If A is a w.o.s of first kind i.e., a w.o.s formed by one-element species, where these elements satisfy a decidable equality relation, and finite only ordered sums, then A is decidable, if its summands are decidable species.
(ii) If A is a w.o.s of second kind, then A is semi-decidable i.e., there is an effective answer to the question $\alpha \in A$ and not to the question $\alpha \notin A$, if each summand is a decidable species (MP)[76].

Proof: (i) If A is an one-element species, then A is decidable, since the equality of the elements of the one-element species is decidable. If $A = \bigoplus_{i=1}^{n} A_i$ and each A_i is decidable, then if α is an appropriate object, i.e., an object for which the question $\alpha \in A_i$'s or not is meaningful, then we apply the effective method of A_i to α. Since A_i are finite in number, then the above procedure is effective to the whole of A.
(ii) In case $A = \bigoplus_{i=1}^{\infty} A_i$ the above method is effective only regarding the answer to the question $\alpha \in A$, and only if we accept as "effective" the following procedure. If α is an object for which the question $\alpha \in A_i$'s or not is meaningful, then if α turns out to be in A, then that will be known in finite time, since at some i the question $\alpha \in A_i$ will be answered positively. If α is not in A, then we cannot be sure of it at any finite time, since checking $\alpha \in A_i$ or not takes infinite time.
If one accepts the effectiveness of the above method, then he accepts the existence of

[76]If this result is accepted, then one has to accept **Markov's principle (MP)**

$$\neg\neg\exists n A(n) \Rightarrow \exists n A(n).$$

Intuitionistically though,this semi-decidability is not accepted for the same reasons MP is not accepted, since "waiting"-arguments are not intuitionistically accepted.

that i without being able to determine a bound of time in which this i will be found. Intuitionistically it is impossible not to exist such an i, but strong intuitionistic existence of i is not justified, since it cannot be specified a finite interval in which i is certainly found.$\diamond$

Next proposition is a partial inverse.

Proposition 10.6: If w.o.s $A = \bigoplus_{i=1}^{n} A_i$, or $A = \bigoplus_{i=1}^{\infty} A_i$, is decidable, then each w.o.s A_i is decidable.

Proof: Let α be an appropriate object and A_i a fixed summand. Applying the effective method of A on α we get $\alpha \notin A$ or $\alpha \in A$. If $\alpha \notin A$, then $\alpha \notin A_i$, since if $\alpha \in A_i$, then $\alpha \in A$, which is absurd. If $\alpha \in A$, then there is some j, such that $\alpha \in A_j$. A_j is unique, since A_i's are disjoint. If $i = j$, then $\alpha \in A_i$, while if $i \neq j$, then $\alpha \notin A_i$.
Thus, in any case we decide effectively if α belongs to A_i or not.$\diamond$

Proposition 10.7: If A is a w.o.s, then A is a finite or countably infinite species. Moreover, a w.o.s of first kind is finite, while a w.o.s of second kind is countably infinite.

Proof: If $A = \{\alpha\}$, then it is finite, while if $A = \bigoplus_{i=1}^{n} A_i$ and each A_i is finite or countably infinite, the same is true for A. If $A = \bigoplus_{i=1}^{\infty} A_i$, then A is countably infinite as a countable union of (disjoint) countably infinite or finite species.$\diamond$

Proposition 10.8: If B is a decidable subspecies of a decidable w.o.s A, then B is also a w.o.s.

Proof: If $A = \{\alpha\}$, then its only subspecies are itself and the empty species. If $A = \bigoplus_{i=1}^{n} A_i$ and B is a decidable subspecies of A, then the decidable subspecies B_i of A_i are defined, where $B_i = B \wedge A_i$. Obviously, $B = \bigoplus_{i=1}^{n} B_i$. Likewise, if $A = \bigoplus_{i=1}^{\infty} A_i$.$\diamond$

A *bar* B for a spread M is a species of finite M-sequences such that, each infinite M-sequence α has an initial segment in B or *hits the bar* i.e.,

$$\forall \alpha(\alpha \in M)(\exists n)(n_\alpha \in B).$$

The above defining property of a bar B does not determine a new species but it presupposes an already constructed species B for which there is a constructive proof of that property. It is only then that we can safely say that B is a bar for a spread M.
A trivial example of a bar for a spread M is the species of all finite M-sequences, which we call the *universal bar* for M. If $\varphi : M \to \omega$ is an M-intuitionistic function, the species B_φ of critical for φ nodes is called *the* φ*-bar*, or *the bar of* φ, and it is the species of those M-nodes which activate φ^*.
A φ-bar is **monotone** i.e.,

$$u \in B_\varphi, u \prec v \Rightarrow v \in B_\varphi.$$

Since φ is defined on $[M]$, each M-sequence cuts B_φ in each node critical for φ. B_φ contains all the information on φ, since it hods together those nodes which activate φ^*.
As Dummett accurately remarks (in [Dummett 2000] p.49)

> [... although "every element of a spread is an infinite choice sequence, we can nevertheless get the effect of all paths terminating by supposing that there is some species B of finite sequences which bars the vertex".]

To show that a species of M-nodes is a bar is, generally, far from trivial. In order to do that we have to construct *for each* α in M a natural number N, which is the length of the initial segment of α in B. In that way an intuitionistic property $R(\alpha, n)$ is constructed such that $R(\alpha, n)$ iff $n_\alpha \in B$. $R(\alpha, n)$ is decidable i.e., we know that n satisfies $R(\alpha, n)$ or not iff B is decidable. A *decidable* bar is one for which there is a method to say in finite time whether a finite M-sequence is in the bar or not. The decidability of B is not entailed in its definition. The universal bar is decidable, since Λ_M is decidable. Also, a φ-bar is decidable, since the mechanism of φ^* decides if an M-node is critical or not. Conversely, if B is a decidable and monotone M-bar and $\varphi' : B \to \omega$, such that:

$$\text{if } u, v \in B,\ u \preceq v \Rightarrow \varphi'(u) = \varphi'(v),$$

then φ' is extended to a function $\varphi : M \to \omega$, where the value of a sequence α under φ is the value $\varphi'(u)$, where α cuts B in u. In that way B becomes the bar of φ.
Decidability of a bar B is, in our view, connected to a "serious" knowledge of species B, and it is not surprise to us that lack of decidability of a bar B has non-desired consequences (see Kleene's counterexample to Brouwer's bar theorem in Paragraph 13).
The non-trivial part of proving a species of M-nodes to be a bar is that we have to find a uniform way of understanding $[M]$ in order to show the bar property for *each* sequence in $[M]$. There is an infinite character in the expression "each sequence in $[M]$" which has to be grasped finitely.
An M-bar B is called *thin* iff B contains only the elements necessary to be a bar i.e.,

$$(u \in B) \wedge (v \prec u) \Rightarrow v \notin B.$$

Hence, in a thin bar there is no pair of comparable nodes, and by the intuitionistic interpretation of quantifiers in the definition of a bar, there is an intuitionistic function $\varphi : M \to \omega$ defined by $\alpha \mapsto l(u)$, where u is the unique initial segment of α which cuts B. If ω^k is the species of finite sequences of length k, then ω^k is a thin bar for the universal spread ω^ω. Also, if $\varphi : M \to \omega$ is an intuitionistic function, the species B_{0_φ} of least critical nodes for φ^* i.e., of those nodes which activate φ^* for the first time, is a thin bar for M.

If B is a bar for spread M, a *sub-bar* C of B, $C \prec B$, is a subspecies of B which is also a bar for M. For example, the thin bar B_{0_φ} is a sub-bar of B_φ.

Proposition 10.9: A decidable bar M for spread M includes always a distinguished thin decidable sub-bar B_0, which we call the *least* thin sub-bar of B. Also, it is impossible for two thin sub-bars of B to be proper sub-species of each other.

Proof: If u is an element of bar B, we can find in finite time, due to the decidability of B, the ancestor u_0 of u which has the least length (≥ 1) among those ancestors of u which belong to B. The species B_0 of those nodes u_0 is a thin sub-bar of B.
If B_1, B_2 two thin sub-bars of bar B such that $B_1 \prec B_2$, and $u \in B_2\ \wedge\ u \notin B_1$, then u has an infinite M-extension α which cuts B_1 at some n_α. Obviously, $n_\alpha \neq u$, otherwise u would belong to B_1 too. Hence, in B_2 the comparable nodes u, n_α coexist, which is absurd.$\diamond$

The universal bar of the spread ω^ω has the species ω^k as infinite in number thin bars, none of which is, trivially, sub-species of each other. Its distinguished thin bar is ω^1.

11. Brouwer's proof of Fan theorem through Bar theorem. BFT is invalid in case a spread which is not a fan is considered. E.g., the intuitionistic function $\varphi : \omega^\omega \to \omega$ defined by

$$\varphi(\alpha) = \alpha(\alpha(0))$$

violates BF(ω^ω), since $\alpha(0)$ can be any natural number n, therefore

$$\varphi(\alpha) = \alpha(n).$$

No N-initial segment though, of any sequence α can include the arbitrarily large term $\alpha(n)$. This counterexample is extended to any spread which is not a fan (see Proposition 12.3).

Hence, the proof of BFT has to reveal that fundamental difference between a fan and a spread which is not a fan, which is responsible for the validity of BFT on fans and not on non-fans spreads.

Brouwer's proof of BFT has a special feature, which is responsible for not being assimilated in his era. This special feature has to do with the use of the intuitionistic interpretation of implication in the proof itself.

With respect to BHK interpretation of logical constants the proof of an implication

$$P \Rightarrow Q$$

is interpreted as a constructive method transforming a proof of P to a proof of Q. I.e., in contrast to its classical interpretation, $P \Rightarrow Q$ is interpreted as follows:

If $K(P)$ is a supposable construction-proof of P, then $P \Rightarrow Q$ is a constructive method transforming $K(P)$ to a construction-proof $K(Q)$ of Q.

Hence, hypothesis P in $P \Rightarrow Q$ is intuitionistically richer in content than in classical $P \Rightarrow Q$. P doesn't only express fact P, but also construction $K(P)$, without which P is only a linguistic hypothesis.

Generally, a proof of an intuitionistic theorem, which is, as any theorem, an implication $P \Rightarrow Q$, takes into account only the fact P, without intervention of $K(P)$ in it. In the proof of BFT though, Brouwer employees $K(P)$. The structure of supposable $K(P)$ is essential to the derivation of $K(Q)$. There is a certain ambiguity though, in the term supposable construction $K(P)$ of P.

$K(P)$ is

(I) either a construction which has already been done, or

(II) a construction which can be done, but not necessarily already done.

The difference between these two interpretations of $K(P)$ is actually the object of a disagreement between Freudenthal and Heyting (see [Petrakis 2010]).

Brouwer's proof of BFT interprets $K(P)$ as in (II) and proves Q in the implication $P \Rightarrow Q$ of BFT analyzing the supposable construction $K(P)$.

Brouwer proves BFT through his Bar theorem. We need to give some definitions first. If B is a decidable bar for a spread M and B_0 is the distinguished thin sub-bar of B, a node u is called *secured* with respect to M iff

$$(\exists v \preceq u, \quad v \in B_0) \vee u \notin M$$

i.e., if we know with certainty the relation of u to B_0. Either u has an ancestor in B_0, which is compatible to $u \notin M$ i.e., the above disjunction is not exclusive, or, knowing that $u \notin M$, no descendant of u cuts B_0, since otherwise u would be an M-node as an ancestor of an M-node. By decidability of B_0 and Λ_M the secured property of a node is decidable. Also, a node v is called *non-secured* iff $v \in M$ but it doesn't cut B_0 yet. If Δ is a decidable species of M-nodes, then an M-node u is called Δ-*securable* iff each infinite M-sequence extending u cuts Δ. I.e., if $\alpha \succ u$, then there is a natural number n, such that $n_\alpha \in \Delta$.

Proposition 11.1: If B is a species of M-nodes, where M is a spread, the following are equivalent:
(i) B is a decidable bar for M.
(ii) Root $<>$ is B-securable.

Proof: $(i) \Rightarrow (ii)$ Since B is a decidable bar for M, if α is any M-sequence extending $<>$, i.e, any M-choice sequence, then α cuts B, by the definition of a bar.
$(ii) \Rightarrow (i)$ If $<>$ is B-securable, then each infinite M-sequence cuts B, therefore B is a bar.$\diamond$

Obviously, if species Δ is a bar, it has no meaning to talk about a non B-securable node u.

Brouwer's Bar theorem is the following proposition with the addition of the decidability of bar. As Kleene showed (see Paragraph 13) the decidability condition, that Brouwer didn't mention, is necessary. Decidability condition is also necessary for intrinsic to the understanding of the concept of species reasons.

Brouwer's Bar theorem (BBT_1): If B is a decidable bar for a spread M (this is hypothesis P of BBT_1), then B contains a well-ordered thin sub-bar (this is conclusion Q of BBT_1).

By Proposition 11.1, Brouwer proves the following version of Bar theorem:

Brouwer's Bar theorem (BBT_2): If B is a decidable species of M-nodes, where M is a spread, such that the root is B-securable (this is hypothesis P' of BBT_2), then B contains a well-ordered thin sub-bar (this is conclusion Q of BBT_2).

The essence of Brouwer's proof of BBT_2 lies in his effort to give a constructive meaning to the universal quantification on infinite choice sequences. Hypothesis P' expresses the fact that each infinite M-choice sequence cuts B i.e., it has the form: $(\forall\alpha \in M)$ $A(\alpha)$. Of course A is an intuitionistic predicate, since α cutting B is activated by an initial segment of α. Since BBT_2 is actually an implication, its BHK-interpretation is the following:

If $R_{<>}$ is a supposable constructive proof of B-securability of the root $<>$, then by $R_{<>}$ a well-ordered thin bar b_0 is constructed.

Since the structure of proof $R_{<>}$ is essential to the proof of BBT_2, and since there are in general indefinite number of possible proofs $R_{<>}$, Brouwer is forced to make a fundamental assumption on the structure of such a proof $R_{<>}$ in order to tame their initial indefinite multiplicity. This fundamental assumption of Brouwer was named by Martino and Giaretta, in [Martino, Giaretta 1979], as Brouwer's dogma:

Brouwer's Dogma (BD): If B is a bar for a spread M, then a proof R_u, of the fact "u

is B-securable", can be reduced to a *canonical proof* (c.p.)[77], where *only* the following kinds of inference occur:

$$\frac{u\ secured}{u\ securable},\quad \eta\text{-inference}$$

$$\frac{(a_1,a_2,...,a_n)\ securable}{(a_1,a_2,...,a_n,k)\ securable},\quad \zeta\text{-inference}$$

$$\frac{(a_1,a_2,...,a_n,1)\ securable,\ (a_1,a_2,...,a_n,2)\ securable,....}{(a_1,a_2,...,a_n)\ securable},\quad \digamma\text{-inference}$$

An **η-inference** expresses the securability of a secured node u. An η-inference is trivially correct, since, if u has already cut B, then each infinite M-extension of u has cut B too, while if $u \notin M$, then there is no such thing as an infinite M-extension of u, and securability of u holds in a trivial way.
A **ζ-inference** expresses the securability of a node $(a_1, a_2, ..., a_n, k)$ if its immediate ancestor $(a_1, a_2, ..., a_n)$ is securable. Clearly, it is also a correct inference, since each infinite M-extension of $(a_1, a_2, ..., a_n, k)$ is an infinite M-extension of $(a_1, a_2, ..., a_n)$, therefore it cuts B at some point of its generation.
A **$\digamma$-inference** expresses the securability of a node $(a_1, a_2, ..., a_n)$ when each immediate descendant $(a_1, a_2, ..., a_n, k)$, $k \in \omega$, of $(a_1, a_2, ..., a_n)$ is securable. A $\digamma$-inference is a correct inference, since each infinite M-extension α of $(a_1, a_2, ..., a_n)$ is an extension of some $(a_1, a_2, ..., a_n, k)$, hence, if $(a_1, a_2, ..., a_n, k)$ is securable, then α cuts B at some point. In a $\digamma$-inference k in descendants $(a_1, a_2, ..., a_n, k)$ is any natural number. Even if the node $(a_1, a_2, ..., a_n, k)$ is not M-accepted, the securability of $(a_1, a_2, ..., a_n, k)$ is derived from an η-inference. The inclusion in a $\digamma$-inference of all nodes $(a_1, a_2, ..., a_n, k)$ and not only of the M-accepted nodes $(a_1, a_2, ..., a_n, k)$ permits the formulation of a $\digamma$-inference without knowing the sequence of natural numbers k which extend the node $(a_1, a_2, ..., a_n)$.
Essential to a $\digamma$-inference is the infinity of its premises, which is understood though, from the intuitionistic point of view. I.e., through an effective way to find a proof of the fact that $(a_1, a_2, ..., a_n, k)$ is securable, for each k.
A stronger interpretation of the above effective method is to understand dots in a $\digamma$-inference as a *common method* of proving its premises i.e., as a common method generating each of these subproofs of the securability of $(a_1, a_2, ..., a_n, k)$. In this case the knowledge of the sequence of k's extending $(a_1, a_2, ..., a_n)$ is needed.
If the spread M in question is a fan, then a $\digamma$-inference has a more concrete structure, since we take into account only those finite in number nodes $(a_1, a_2, ..., a_n, k_i$ which extend $(a_1, a_2, ..., a_n)$, and which can be found in finite time by the definition of Λ_M. In the fan case a $\digamma$-inference then becomes:

$$\frac{(a_1,a_2,...,a_n,k_1)\ securable,\ (a_1,a_2,...,a_n,k_2)\ securable,...,(a_1,a_2,...,a_n,k_m)\ securable}{(a_1,a_2,...,a_n)\ securable},$$

where $k_1, k_2, ..., k_m$ are the immediate successors of $(a_1, a_2, ..., a_n)$[78].
While η, ζ and $\digamma$-inferences are correct inferences, what is far from trivial in BD is Brouwer's assumption that these are the *only* inferences on which one can count in order to construct proof R_u. This crucial point in Brouwer's argumentation is discussed

[77]A formal definition of a canonical proof is given in Paragraph 14.

[78]We call a natural number k immediate successor of a node $(a_1, a_2, ..., a_n)$ if the node $(a_1, a_2, ..., a_n, k)$ is an immediate successor node of $(a_1, a_2, ..., a_n)$.

from the epistemological and mathematical point of view in Paragraph 14. It suffices to say here that Brouwer considered BD an intuitionistic truth of which he never found a complete justification. The "evidence" of BD is of course questioned in the literature. An informal and partial justification of BD consists in the following arguments:

(I) At first we only know decidable species B, and we immediately get securability of all nodes which belong to B or extend nodes which belong to B, through η-inferences. (II) Nodes are connected with successive applications of immediate ancestor and descendant relations. It seems that the only way to connect the known securability of a node u_1 with the in question securability of a node u_2, is to go from u_1 to u_2 through successive immediate descendants of u_1 i.e., through ζ-inferences, or through successive immediate ancestors of u_1 i.e., through F-inferences.

Some definitions are needed before starting proving BBT_2.

If B is a bar for spread M, *a securable node u has the well-ordering property for nodes* (w.o.p.n) iff the thin bar $B_0^u \preceq B_0$ which bars exactly the M-sequences which extend u is a w.o.s. I.e., subspecies B_0^u of B_0 which lies "in front" of u is a w.o.s.
If a node u satisfies w.o.p.n, then the conclusion of BBT_2 is "locally" established, since B_0^u is locally thin sub-bar and also a w.o.s. We note that since B in BBT_1 is a decidable bar, B_0^u is a decidable subspecies of B_0. And this is the case because B is decidable, therefore B_0 is decidable, and the question if $v \in B_0$ extends u or not is of course decidable.
A subproof R of proof R_u has the well-ordering property of canonical proofs (w.o.p.p) iff the *conclusions* in R i.e., each node v the securability of which is proved in R, has the w.o.p.n. for nodes. Obviously, w.o.p.p expresses a more global approximation of BBT_2's conclusion. Not only for one node v, but for each node v proven to be secure in R, we know that thin B_0^v is a w.o.s.
A subproof R of proof R_u has the preservation property of canonical proofs (p.p.p) iff each conclusion v in R has the w.o.p.n, whenever each node w the securability of which is a premiss in some inference in R has the w.o.p.n. I.e., a subproof R satisfying p.p.p preserves the w.o.p.n from its premisses nodes to its conclusion nodes.
Then, Brouwer proves the following propositions:

Proposition 11.2: R_u has the preservation property of canonical proofs.

Proof: We need to show that each conclusion in proof R_u of the securability of node u has the w.o.p.n, whenever each node in some inference in R_u has the w.o.p.n. Because of BD, we prove this for the three only possible kinds of inferences in canonical proof R_u.
(I) If an η-inference,

$$\frac{v\ secured}{v\ securable},$$

has been used in R_u and the node v of its hypothesis satisfies the w.o.p.n, then the conclusion node trivially satisfies the w.o.p.n, since the conclusion node is v again.
(II) If a ζ-inference,

$$\frac{(a_1,a_2,...,a_n)\ securable}{(a_1,a_2,...,a_n,k)\ securable},$$

has been used in R_u, and by hypothesis the subspecies $B_0^{(a_1,a_2,...,a_n)}$ of the thin bar B_0 which lies in front of $(a_1, a_2, ..., a_n)$ is a w.o.s, we want to prove that the species

$B_0^{(a_1,a_2,...,a_n,k)}$ in front of $(a_1, a_2, ..., a_n, k)$ is also a w.o.s. The implication

$$(*) \qquad B_0^{(a_1,a_2,...,a_n)} \ w.o.s \Rightarrow B_0^{(a_1,a_2,...,a_n,k)} \ w.o.s$$

is proved inductively using IWOS.

The interesting case of $(*)$ is that node $w = (a_1, a_2, ..., a_n)$ is securable and not secured, but we first check the secured case.
If w **is secured**, then,
(i) w has an ancestor w_0 in B_0: then, $w \frown k$ is also secured, therefore securable, through an η-inference.
The same thought applies if w belongs to B_0: in the first case $B_0^w = \{w_0\}$, where B_0^w is "in front" of w directed to the root, while in the second, $B_0^w = \{w\}$. In both cases, B_0^w is an one element w.o.s and correspondingly $B_0^{w \frown k} = B_0^w = \{w_0\}$, or $B_0^{w \frown k} = B_0^w = \{w\}$ i.e., $B_0^{w \frown k}$ is a w.o.s.
(ii) $w \notin M$: Then, of course, $w \frown k \notin M$ too. Since B_0^w is the empty species, implication $(*)$ holds trivially.
Suppose now that node w **is securable and not secured**. Then, subspecies B_0^w is in front of w, and it is a w.o.s.
(i) If $B_0^w = \{w_0\}$, then $B_0^{w \frown k} = B_0^w = \{w_0\}$ i.e., $B_0^{w \frown k}$ is an one-element w.o.s.
(ii) If $B_0^w = \bigoplus_{i=1}^m A_i$, where A_i are w.o.s, then

$$B_0^{w \frown k} = \bigoplus_{i=1}^{m} \Gamma_i,$$

where

$$\Gamma_i = A_i \wedge B_0^{w \frown k}.$$

Since B_0^w is a decidable w.o.s, then by Proposition 10.6, each A_i is decidable, hence, each Γ_i is decidable too, as the conjunction of two decidable species. Hence, each Γ_i, as a decidable subspecies of decidable w.o.s A_i, is also a w.o.s, by Proposition 10.8. Then, $B_0^{w \frown k}$ is a w.o.s, as the finite ordered sum of w.o.s Γ_i.
(iii) If $B_0^w = \bigoplus_{i=1}^\infty A_i$, where A_i are w.o.s, then $B_0^{w \frown k} = \bigoplus_{i=1}^m \Gamma_i$, where $\Gamma_i = A_i \wedge B_0^{w \frown k}$ and $B_0^{w \frown k}$ is a w.o.s, as the infinite ordered sum of w.o.s Γ_i, using exactly the same line of thought as in case (ii).
(III) If a F-inference

$$\frac{(a_1,a_2,...,a_n,1) \ securable, \ (a_1,a_2,...,a_n,2) \ securable,....}{(a_1,a_2,...,a_n) \ securable}$$

has been used in R_u, and by hypothesis, the subspecies $B_0^{(a_1,a_2,...,a_n,k)}$ of the thin bar B_0 which lies in front of $(a_1, a_2, ..., a_n, k)$ is a w.o.s, for each k, we need to prove that the species $B_0^{(a_1,a_2,...,a_n)}$ in front of $(a_1, a_2, ..., a_n)$ is also a w.o.s. As in the general case of a F-inference, we suppose a uniform method of proof that each node $(a_1, a_2, ..., a_n, k)$ is securable, we also suppose a uniform method of proof that each species $B_0^{(a_1,a_2,...,a_n,k)}$ is a w.o.s.
If again we set $w = (a_1, a_2, ..., a_n)$, we examine the following cases:
(i) If node $w \frown k$ is securable because it is secured, then either
$w \frown k$ has an ancestor in B_0, hence $B_0^{w \frown k} = B_0^w = \{w_0\}$, which is an one-element w.o.s and there is nothing else to prove, or $w \frown k$ belongs to B_0, hence $B_0^{w \frown k} = \{w \frown k\}$, or

$w \frown k \notin M$, hence there are no nodes of B_0 in front of $w \frown k$, otherwise $w \frown k \in M$. In this trivial case we set $B_0^{w \frown k} = \{k\}$, to avoid $B_0^{w \frown k}$ seen as an empty species.
(ii) Suppose now that node $w \frown k$ is securable without being secured.

We prove now that all these possible w.o.s $B_0^{w \frown k}$ are disjoint to each other i.e., if $w \frown k$ and $w \frown \lambda$ two immediate successor nodes of w, such that $k \neq \lambda$, then

$$(**) \qquad B_0^{w \frown k} \wedge B_0^{w \frown \lambda} = \emptyset,$$

where $\emptyset$ denotes the empty species.
If $B_0^{w \frown k}$ and $B_0^{w \frown \lambda}$ are one-element species, then $B_0^{w \frown k} = \{k\}$ or $B_0^{w \frown k} = \{w \frown k\}$ and $B_0^{w \frown \lambda} = \{\lambda\}$ or $B_0^{w \frown \lambda} = \{w \frown \lambda\}$. In any case then, $(**)$ is satisfied. Also two nodes of type (ii), which cannot be one-element species, satisfy $(**)$, since a node of B_0 extending $w \frown k$ cannot also extend $w \frown \lambda$ and vice versa.
Thus, a node of B_0 in front of w is in front of some $w \frown k$, for some unique k.
Moreover, given an element of B_0 we can effectively find in which $B_0^{w \frown k}$ is included. I.e.,

$$B_0^w \preceq \bigvee_{k=1}^{\infty} B_0^{w \frown k},$$

and $\preceq$ is explained by the existence of all those k for which $w \frown k \notin M$ and, obviously, $B_0^{w \frown k} = \{k\}$ is not a node of B_0 in front of w. Thus, B_0^w acquires a w.o.s structure, since *it can be seen* as the following ordered sum

$$B_0^w = \bigoplus_{\lambda=1}^{(\infty)} B_0^{w \frown \lambda},$$

where the (possibly) infinite species $B_0^{w \frown \lambda}$ are not one-element species of natural numbers i.e., only subspecies of B_0 in front of immediate successors $w \frown \lambda$ which belong to M are included to this ordered sum. Thus, we proved for all three kinds of inferences the preservation property of R_u.$\diamond$

It is clear that without BD, i.e., without a determination of all possible inferences in a canonical proof, it is impossible to provide the above inductive proof of preservation property of R_u.

Proposition 11.3: R_u has the well-ordering property of canonical proofs.

Proof: We need to show that each node v the securability of which is proved in R_u has the w.o.p.n i.e., the species B_0^v of those nodes of B_0 in front of v is a w.o.s.

R_u necessarily starts from some η-inferences, such that "going up" afterwards through ζ-inferences and mainly "going down" through F-inferences, we reach the securability of node u.

This is the interesting case, since if node u is "above" B_0, then R_u is just an η-inference. When the securability of a node v is proved in R_u through an η-inference though, species B_0^v in front of v is an one-element species, hence a w.o.s, therefore v has the w.o.p.n. Since, by Proposition 11.2, proof R_u has the preservation property, the w.o.p.n is transferred from the nodes-premisses in R_u to nodes-conclusions in R_u.$\diamond$

The main argument of the above proof contains a non-trivial point, from the intuitionistic point of view, discussed in Paragraph 15.

Proposition 11.4: In proof R_u the node u has the well-ordering property for nodes.

Proof: Since the securability of u is proved in proof R_u and R_u has the w.o.p.p, then its final conclusion u has the w.o.p.n.$\diamond$

Proposition 11.5 Brouwer's Bar theorem (BBT_2): If B is a decidable species of M-nodes, where M is a spread, such that the root $<>$ is B-securable, then B contains a well-ordered thin sub-bar.

Proof: We actually prove that the distinguished thin sub-bar B_0 of B is a well-ordered species. Since hypothesis of BBT_2 is a supposable proof of the securability of root $<>$, then, by Brouwer's dogma, there is a canonical proof $R_{<>}$ of the securability of $<>$. Then, by Proposition 11.4, in case $u = <>$, node $<>$ has the w.o.p.n meaning that, the species of nodes $B_0^{<>}$ in front of $<>$ is a w.o.s. Since

$$B_0^{<>} = B_0,$$

B_0 is a w.o.s.$\diamond$

We can give an exact description of the well-ordering of B_0.

Proposition 11.6: The well-ordering of B_0, under the hypotheses of BBT_2, is the lexicographic ordering.

Proof: Let $u = (a_1, a_2, ..., a_n)$ and $v = (b_1, b_2, ..., b_m)$ two different elements of B_0. Since B_0 is a thin bar, it is impossible that u, v are extensions of one another. Hence, there is some index $i \geq 1$ for which $a_i \neq b_i$ for the first time. In lexicographic ordering if $a_i > b_i$, then $u \succ v$, while if $a_i < b_i$, then $u \prec v$.
Let w is the maximum common segment of u, v i.e., $l(w) = i - 1$.
If $k = a(i) < b(i) = \lambda$, then

$$B_0^{(w)} = B_0^{(w\frown 1)} \oplus B_0^{(w\frown 2)} \oplus ... \oplus B_0^{w\frown k} \oplus ...B_0^{w\frown\lambda} \oplus ...,$$

where $B_0^{(w\frown j)}$ is in front of $w \frown j$, if $w \frown j$ is M-accepted. Since $u \in B_0^{w\frown k}$ and $v \in B_0^{w\frown\lambda}$, $u \prec v$, exactly as in the lexicographic ordering.$\diamond$

For example, if M is the universal spread and the nodes $(2,2), (2,5,1), (2,6,2)$ cut the bar B_0, then $B_0^{(2,2)} = \{(2,2)\}$, $B_0^{(2,5,1)} = \{(2,5,1)\}$ and $B_0^{(2,6,2)} = \{(2,6,2)\}$. Since,

$$B_0^{(2)} = B_0^{(2,1)} \oplus B_0^{(2,2)} \oplus ... \oplus B_0^{(2,5)} \oplus B_0^{(2,6)} \oplus ...,$$

by the definition of an infinite ordered sum, we get that

$$(2,2) \prec (2,5,1) \prec (2,6,2),$$

since $(2,2) \in B_0^{(2,2)}$ $(2,5,1) \in B_0^{(2,5)}$ and $(2,6,2) \in B_0^{(2,6)}$. The above ordering is exactly the lexicographic one.
Of course, if we had defined the lexicographic ordering on B_0, we wouldn't get its well-ordering as this has been defined intuitionistically. Lexicographic ordering on its own does not give some information on the structure of $B_0^{(a_1,a_2,...,a_n)}$. Even in a classical setting, lexicographic ordering is not necessarily a well-ordering. E.g., lexicographic ordering is defined on decimal expansions of real numbers, which is not though, a well-ordering. Such a well-ordering is established only through the axiom of choice.

We now complete Brouwer's proof of BFT through BBT_2, showing how the hypothesis of fan is crucial to its proof.

Proposition 11.7 Brouwer's Fan theorem (BFT): If T is a fan and $\varphi : T \to \omega$ an intuitionistic function, then there exists a natural N, such that, for each infinite T-sequence α, its value $\varphi(\alpha)$ is determined by its initial segment N_α.

Proof: By the definition of φ, B_φ is a decidable species therefore, a decidable bar. By BBT_1, B_{φ_0} is a well-ordered species.
If u is node of T that has not yet cut B_{φ_0}, then the species $B^u_{\varphi_0}$ is a decidable sub-species of B_{φ_0}, since properties $v \in B_{\varphi_0}$ and $v \succeq u$ are both decidable. Hence, by Proposition 10.8, $B^u_{\varphi_0}$ is also a w.o.s.
If $u \frown k_1, u \frown k_2, ..., u \frown k_n$ are the immediate successor nodes of u, then

$$B^u_{\varphi_0} = \bigvee_{i=1}^{n} B^{u \frown k_i}_{\varphi_0}.$$

Since $B^{u \frown k_i}_{\varphi_0}$ are mutually disjoint, and also, as we argued for $B^u_{\varphi_0}$ already, it is a w.o.s, then

$$B^u_{\varphi_0} = \bigoplus_{i=1}^{n} B^{u \frown k_i}_{\varphi_0}.$$

If $u \frown k_i \in B_{\varphi_0}$, then $B^{u \frown k_i}_{\varphi_0}$ is an one-element w.o.s, while if not, then $B^{u \frown k_i}_{\varphi_0}$ is a finite direct sum of w.o.s. Going on like that we reach B_{φ_0} i.e., the corresponding constituent w.o.s are one-element w.o.s, and all intermediate species are finite and finally $B^u_{\varphi_0}$ is a finite species.
If we take $u = <>$, then

$$B^{<>}_{\varphi_0} = B_{\varphi_0},$$

which is also finite. Thus, B_{φ_0} has a node of maximum length N, the global bound of BFT.
In the above proof we considered $<>$ not to have cut B. Otherwise, $<>$ belongs to B and BFT is trivially true.$\diamond$

If $BBT_1(\omega^\omega)$ denotes Brouwer's bar theorem on the universal spread, then $BBT_1(\omega^\omega)$ entails BBT_2.

Proposition 11.8: $BBT_1(\omega^\omega) \Rightarrow BBT_1$.

Proof: If B_M is a decidable bar for a spread M, then B_M is extended to a decidable bar B on ω^ω, where

$$B = B_M \vee M',$$

where M' is the species of M-unaccepted nodes. Decidability of B_M and Λ_M entail decidability of B. B is a bar, since each infinite sequence of M is barred by B_M, while each infinite sequence outside M is barred by M'. By hypothesis, B has a well-ordered thin sub-bar B_0. Then,

$$B_0^M = B_0 \cap B_M$$

is well-ordered decidable thin sub-bar of B_M, since:
(i) B_0^M is a sub-bar of B_M, since it is by its definition a subspecies of B_M and each M-sequence cuts B_0 at some node of B_M.

(ii) B_0^M is decidable as the conjunction of two decidable species.
(iii) B_0^M is thin, since if $u \in B_0^M$ and $v \prec u$, then $v \notin B_0$, hence, $v \notin B_0^M$ too.
(iv) B_0^M is well-ordered, since it is a decidable subspecies of a w.o.s (Proposition 10.8).
B_0 includes one-element species $\{(k)\}$, where (k) is an 1-sequence which is not Λ_M-accepted, and species B_0^M.$\diamond$

12. Brouwer's Uniform Continuity theorem. As van Dalen remarks[79], Brouwer's results in BIA are consequences of BFT rather than direct applications of BBT. We study here some fundamental consequences of BFT, especially on intuitionistic Functions.

If $\Phi : M \to \omega^\omega$ is an M, ω^ω-Function, then the species $\Phi(M)$ of sequences $\Phi(\alpha)$, where $\alpha \in M$, is not, generally, a spread (Proposition 12.4). If though, M is a fan, then $\Phi(M)$ is not only a spread, but also a fan. This fact is a consequence of the general validity of BFT on fans only.

Proposition 12.1: If T is a fan and $\Phi : T \to \omega^\omega$ is an intuitionistic Function, then species $\Phi(T)$ is not only a spread, but also a fan.

Proof: (I) First we show that $\Phi(T)$ is a spread.
In order to show that we show how a decidable law $\Lambda_{\Phi(T)}$, which generates the choice sequences of $\Phi(T)$, is defined.
An 1-sequence (n) is $\Lambda_{\Phi(T)}$-accepted iff n is the first term of some sequence $\Phi^*(u)$, where u is Λ_T-accepted.
In order $\Lambda_{\Phi(T)}$ be decidable, this acceptance must be decided in finite time. I.e., all T-accepted nodes u which activate Φ^* for the first time must be checked in finite time. Since T is a fan this is possible, because, if we define the function $\varphi_1 : T \to \omega$,

$$\varphi_1(\alpha) = [\Phi(\alpha)]_1 = [\Phi^*(u)]_1,$$

where $[\Phi(\alpha)]_1$ is the first element of sequence $\Phi(\alpha)$ and u the first initial segment of α activating Φ^*, then φ_1 satisfies BFT, therefore, there is some index N_1 for which all T-sequences activating Φ^* for the first time are of length $\leq N_1$. Since those sequences in a fan are finite, the above check of law $\Lambda_{\Phi(T)}$ for 1-sequences is effective. In the general case $\Lambda_{\Phi(T)}$ works as follows:
A sequence $(b_1, b_2, ..., b_n)$ is $\Lambda_{\Phi(T)}$-accepted iff $(b_1, b_2, ..., b_n)$ is the n-initial segment some node $\Phi^*(u)$, where u is Λ_T-accepted.
$\Lambda_{\Phi(T)}$ is again decidable through function $\varphi_n : T \to \omega$, where

$$\varphi_n(\alpha) = [\Phi(\alpha)]_n = [\Phi^*(u)]_n,$$

and $[\Phi(\alpha)]_n$ is the n-th term of sequence $\Phi(\alpha)$ and u is the first initial segment of α activating Φ^* such that its image $\Phi^*(u)$ has length $\geq n$.
$\Lambda_{\Phi(T)}$ also allows the ever extension of the nodes it accepts, since Φ^* is not finally constant.
Hence, $\Lambda_{\Phi(T)}$ is a spread law, which generates exactly those sequences of species $\Phi(T)$, since each infinite $\Lambda_{\Phi(T)}$-sequence belongs to $\Phi(T)$, and each sequence of $\Phi(T)$ is $\Lambda_{\Phi(T)}$-generated, since each of its initial segments is $\Lambda_{\Phi(T)}$-accepted.

[79] See [Brouwer 1981] p.101.

(II) We now show that $\Phi(T)$ is also a fan.
If $\Phi^*(u)$ is considered, then each immediate successor node of $\Phi^*(u)$ is an initial segment of the value under Φ^* of some node v which extends u, and v is the first successor node of u for which $\Phi^*(v)$ strictly extends $\Phi^*(u)$. If we show that these sequences v are finite, then the immediate successor nodes of $\Phi^*(u)$ are also finite.
For that we define $\varphi_u : B(u) \to \omega$, where $B(u)$ is the species of those infinite T-sequences which extend u[80], by

$\varphi_u(\alpha) = k$, where k is the length of the initial segment of α for which $\Phi^*(k_\alpha) \succ \Phi^*(u)$.

φ_u is well-defined, since Φ^* is not finally constant and k can be effectively found. Also, species $B(u)$ is a sub-fan of T, with $\Lambda_{B(u)}$ the law which accepts u and each T-sequence extending u. Hence, $B(u)$ is also a fan, therefore, by BFT, there is some N, such that all critical to φ_u nodes have length $\leq N$. Thus, sequences v extending u such that $\Phi^*(v)$ strictly extends $\Phi^*(u)$ for the first time are of length $\leq N$, therefore, they are finite in number.
If w is an initial segment of some $\Phi^*(u)$ without being some $\Phi^*(u')$, we define function $\varphi_w : T \to \omega$, where $\varphi(\alpha)$ is the least length k for which $\Phi^*(k_\alpha) \succ w$, if $\Phi(\alpha) \succ w$ and 0, if $\neg[\Phi(\alpha) \succ w]$. But,

$$\Phi(\alpha) \succ w \vee \neg[\Phi(\alpha) \succ w],$$

since w is a finite node and then φ_w is well-defined. Applying BFT on φ_w we find again that the immediate successor nodes of w are finite in number.$\diamond$

Proposition 12.2: If T is a splitting fan and $\Phi : T \to \omega^\omega$ an intuitionistic $1-1$ Function, then species $\Phi(T)$ is also a splitting fan.

Proof: Let a $\Phi(T)$-node $\Phi^*(u)$. We show that $\Phi^*(u)$ has two non-comparable extension $\Phi^*(u_1)$ and $\Phi^*(u_2)$. By hypothesis, u has two non-comparable T-extensions u_1, u_2. Hence, there are infinite T-sequences α, β extending u_1, u_2. Since Φ is $1-1$, $\Phi(\alpha) \neq \Phi(\beta)$. I.e., at some point of their generation $\Phi(T)$-sequences $\Phi(\alpha), \Phi(\beta)$, extending both $\Phi^*(u)$, split i.e., $\Phi^*(u)$ splits. If node w is an initial segment of a node $\Phi^*(u)$, then w splits, since its extension $\Phi^*(u)$ splits.$\diamond$

As we saw in proof of Proposition 12.1, BFT was necessary in order $\Lambda_{\Phi(T)}$ is decidable. Since fan theorem is violated in case a spread is not a fan, Proposition 12.1 is expected not to hold for them.
Intuitionistically speaking, a spread which is not a fan is a spread for which it is impossible to be a fan, through the BHK-interpretation of negation. A spread M is *positively not a fan* iff we know an M-node u with infinite number of immediate successor nodes.

Proposition 12.3: If M is a spread positively not a fan, then there is an intuitionistic function $\varphi : M \to \omega$, such that there are least critical nodes for φ^* of arbitrary length, therefore the conclusion of BFT doesn't hold for M.

Proof: Let $u = (\alpha_0, \alpha_1, ..., \alpha_{n-1})$ an -node of length n for which we know that it has infinite immediate successors. We then define $\varphi : M \to \omega$ by

$$\varphi(\alpha) = \alpha(\alpha(n)).$$

[80]Classically, $B(u)$ is the clopen basic set of Baire space (see the Appendix).

If we consider only sequences α extending u, we see that numbers $\alpha(n)$ are infinite, hence arbitrary large. Therefore, the length of the initial segments of these sequences α which activate φ^* is arbitrary large too.

Proposition 12.4: If M is a spread positively not a fan, there is an intuitionistic Function $\Phi : M \to 2^\omega$, such that species $\Phi(M)$ is not a spread.

Proof: Let $\varphi : M \to \omega$ be the function of the previous proof. We define $\Phi : M \to 2^\omega$ as follows:
If u is an M-node, non-critical for φ^*, and of length n, then

$$u \overset{\Phi^*}{\mapsto} (\underbrace{1, 1, ..., 1}_{n}),$$

while if M-node u is critical for φ^*, then

$$u \overset{\Phi^*}{\mapsto} (\underbrace{\underbrace{1, 1, ..., 1}_{N-1}, 0, 0, ..., 0}_{n}),$$

where n is the length of u and N is the length of the least critical sequence included in u.
Since there are least critical for φ^* nodes of arbitrary length, any initial segment of constant sequence $\overline{1}$ is the image some non-critical for φ^* -node u under Φ^*. Hence, each initial segment of $\overline{1}$ is the initial segment of a sequence of $\Phi(T)$, while $\overline{1}$ doesn't belong to it. Otherwise, there would be an M-sequence none initial segment of which is critical for φ^*, hence φ couldn't be defined on it. This is impossible, since all of $[M]$ is the domain of definition of φ. Thus, $\Phi(T)$ is not a spread, since a spread always contains a sequence, each initial segment of which is the initial segment of some of its sequences.$\diamond$

The above propositions show how properties of intuitionistic Functions depend on the behavior of intuitionistic functions (we have already seen this in Veldman's proof of continuity of a real Function of Brouwer) and BFT.
A species A of sequences of natural numbers is called *analytic* iff it is the empty species or the image of the unversal spread under an intuitionistic Function Φ i.e., if there is a $\Phi : \omega^\omega \to A$, such that

$$\Phi(\omega^\omega) = A.$$

As we have showed in Proposition 9.2, each spread is analytic species. The analytic species $\Phi(\Theta(\omega^\omega))$, where $\Theta : \omega^\omega \to M$ the retraction Function of Proposition 9.2 on M and M, Φ as in Proposition 12.4, is not a spread.

Proposition 12.5: The image of an analytic species A under some intuitionistic Function is also analytic species.

Proof: If $\Phi : \omega^\omega \to A$ and $\Theta : A \to \Theta(A)$ are intuitionistic Functions, then Function $\Theta \circ \Phi : \omega^\omega \to \Theta(A)$ is defined. $(\Theta \circ \Phi)^*(u) = \Theta^*(\Phi^*(u))$ computes $\Theta \circ \Phi$, which is obviously onto $\Theta(A)$.$\diamond$

Analytic species though, do not behave as classical analytic sets since, for example, the intersection of analytic species is not, in general, analytic species too. As we saw in

Paragraph 5, the intersection of two spreads is not, in general, a spread too, since we defined two spreads, the intersection of which couldn't say if it was the constant zero sequence or the empty spread. Hence, we cannot define an intuitionistic Function from ω^ω to the intersection, since we do not know whether the nodes of naturals must correspond to the empty sequence or to initial segments of $\overline{0}$. Our ignorance regarding the evolution of a spread because of the dependence of spreads on unsolvable mathematical problems is behind the difference between analytic species and analytic sets.

Proposition 12.6: If T is a fan, then there is homeomorphism $\Phi : T \to T_2$, where T_2 is a sub-fan of 2^ω.

Proof: By Proposition 12.1, it suffices to define an invertible Function $\Phi : T \to 2^\omega$, such that $\Phi(T)$ is the sub-fan T_2. If $(a_1, a_2, ..., a_n)$ is a T-node, Φ^* is determined by the following correspondence:

$$(a_1, a_2, ..., a_n) \overset{\Phi^*}{\mapsto} (\underbrace{1, 1..., 1}_{a_1+1}, 0, \underbrace{1, 1..., 1}_{a_2+1}, 0, ..., 0, \underbrace{1, 1, ..., 1}_{a_n+1}).$$

Φ^* is monotone, Function Φ determined by Φ^* is $1-1$, and its inverse is defined on the nodes of T_2 by the inverse law Φ^{-1*} to Φ^*.$\diamond$

BFT on 2^ω has, of course, the following form:

$$BFT(2^\omega) \qquad (\exists N)(\forall\alpha)(\alpha \in 2^\omega)\varphi(\alpha) = \varphi^*(N_\alpha).$$

Proposition 12.7: BFT is equivalent to $BFT(2^\omega)$.

Proof: If T is a fan, then, by Proposition 12.6, there is a homeomorphism $\Phi : T \to T_2$, where T_2 is a sub-fan of 2^ω. Since, there is a retraction Function $\Theta : 2^\omega \to T_2$, $BFT(2^\omega)$ on function $\varphi \circ \Phi^{-1} \circ \Theta$ entails BFT regarding function $\varphi : T \to \omega$. If $\alpha \in T$ and $\beta = \Phi(\alpha)$, then

$$(\varphi \circ \Phi^{-1} \circ \Theta)(\beta) = (\varphi \circ \Phi^{-1})(\beta) = \varphi(\alpha).$$

By $BFT(2^\omega)$, the value $(\varphi \circ \Phi^{-1} \circ \Theta)(\beta)$ is determined by a segment $M_\beta = M_{\Phi(\alpha)}$. But a segment $(\underbrace{1, ..., 1, 0, 1, ..., 1, 0, ..., 0, 1, ..., 1}_{M})$ of length M corresponds to a segment N_α of α, where N is at most $\frac{M+1}{2}$, if M is odd, or $\frac{M}{2}$, if M is even. Thus $N = \frac{M+1}{2}$ is a global bound of φ, since N_α determines M_β, which in turn determines value $\varphi(\alpha)$.$\diamond$

BFT is the most important proposition of BIA, since it proves the remarkable, from the classical point of view, uniform continuity theorem (UCT), according to which an intuitionistic Function $\Phi : [\alpha, \beta] \to \Re_{Br}$ is uniformly continuous, just by being defined on the intuitionistic closed interval $[\alpha, \beta]$.

We follow here Heyting's presentation of the proof of UCT (see [Heyting 1966] pp.46-47), including only what is necessary with respect to properties of order of intuitionistic reals. A condensed proof of UCT can also be found in [Brouwer 1927].

A **canonical real number generator** (c.r.n.) is a sequence $(\frac{\lambda_n}{2^n})_n$, where $\lambda_n \in \mathbb{Z}$, such that

$$|\frac{\lambda_n}{2^n} - \frac{\lambda_{n+1}}{2^{n+1}}| < \frac{1}{2^n}, \qquad (1)$$

for each n. It is easy to see that (1) guarantees that $(\frac{\lambda_n}{2^n})_n$ is a r.n. i.e., an intuitionistic Cauchy sequence.

Proposition 12.8: If q_k is a rational number, then there is a unique rational number $\frac{\lambda_n}{2^n}$, where $\lambda_n \in \mathbb{Z}$, such that

$$|q_k - \frac{\lambda_n}{2^n}| \leq \frac{1}{2^{n+1}}. \quad (2)$$

Proof: $|q_k - \frac{\lambda_n}{2^n}| \leq \frac{1}{2^{n+1}} \Leftrightarrow -\frac{1}{2^{n+1}} \leq \frac{\lambda_n}{2^n} - q_k \leq \frac{1}{2^{n+1}} \Leftrightarrow$

$$2^n q_k - \frac{1}{2} \leq \lambda_n \leq 2^n q_k + \frac{1}{2}.$$

If the ends of the interval $[2^n q_k - \frac{1}{2}, 2^n q_k + \frac{1}{2}]$ are not integers, then λ_n is the unique integer in it, and (2) is satisfied.
If the ends of the interval $[2^n q_k - \frac{1}{2}, 2^n q_k + \frac{1}{2}]$ are integers, then, if $m = 2^n q_k - \frac{1}{2}$ and $m + 1 = 2^n q_k + \frac{1}{2}$, $q_k = \frac{2m+1}{2^{n+1}}$, and if we set $\lambda_n = m$,

$$|q_k - \frac{\lambda_n}{2^n}| = |\frac{2m+1}{2^{n+1}} - \frac{m}{2^n}| = \frac{1}{2^{n+1}}$$

i.e., we get equality.$\diamond$

Proposition 12.9: If α is a r.n., then there is a c.r.n. $(\frac{\lambda_n}{2^n})_n$, such that

$$|\alpha - \frac{\lambda_n}{2^n}| < \frac{5}{8}\frac{1}{2^n}, \quad (3)$$

i.e., α is representable by a c.r.n.

Proof: If $(q_{\alpha(k)})$ is a representation of α, and since there is some n_0 such that $|q_{\alpha(k)} - \alpha| < \frac{1}{2^{n+3}}$, for each $k \geq n_0$, then, using Proposition 12.8, someone could write

$$|\alpha - \frac{\lambda_n}{2^n}| \leq |\alpha - q_{\alpha(k)}| + |q_{\alpha(k)} - \frac{\lambda_n}{2^n}| < \frac{1}{2^{n+3}} + \frac{1}{2^{n+1}} = \frac{5}{8}\frac{1}{2^n}.$$

Also,

$$|\frac{\lambda_n}{2^n} - \frac{\lambda_{n+1}}{2^{n+1}}| \leq |\frac{\lambda_n}{2^n} - \alpha| + |\alpha - \frac{\lambda_{n+1}}{2^{n+1}}| < \frac{5}{8}\frac{1}{2^n} + \frac{5}{8}\frac{1}{2^{n+1}} = \frac{15}{16}\frac{1}{2^n} < \frac{1}{2^n}.$$

Actually, the above argumentation is not intuitionistically correct, since the triangle inequality does not hold, only a variation of it (see Proposition 12.17). We reach the above inequality though, through some of the following propositions. What is needed for the exact formulation of the above proof can be found in the proof of Proposition 12.19. We present this standard simple technique there.$\diamond$

We strengthen a bit now the definitions of equality and order between r.n. given in Paragraph 8. Namely, a real number (generator) (r.n.) is a sequence $(q_{\alpha(n)})$ of rational numbers satisfying

$$(\forall k)(\exists n_{0\alpha})(\forall n \geq n_{0\alpha})\ |q_{\alpha(n)} - q_{\alpha(n+1)}| < \frac{1}{2^{k+1}}.$$

Also,

$$\alpha \approx \beta \Leftrightarrow (\forall k)(\exists n_0)(\forall n \geq n_0)\ |q_{\alpha(n)} - q_{\beta(n)}| < \frac{1}{2^k},$$

and

$$(q_{\alpha(n)}) < (q_{\beta(n)}) \Leftrightarrow (\exists k)(\exists n_0)(\forall n \geq n_0)\ q_{\beta(n)} - q_{\alpha(n)} > \frac{1}{2^k},$$

$$\alpha < \beta \Leftrightarrow (\exists (q_{\alpha(n)}))(\exists (q_{\beta(n)}))\ (q_{\alpha(n)}) < (q_{\beta(n)}).$$

Note that all expected properties of order and of other operations on rationals intuitionistically hold. Unfortunately, the transition of these properties to the intuitionistic reals is not always possible.
It is easy to see that $<$ is compatible to $\approx$ i.e., if $\alpha < \gamma$ and $\alpha \approx \beta$, $\gamma \approx \delta$, then $\beta < \delta$.
In Paragraph 5 we defined the intuitionistic closed interval $[\alpha, \beta]$ as an appropriate spread. Its points were its choice sequences. A tacit assumption of that definition was that $\alpha < \beta$ was considered known, and it is easy to prove that for each $x \in [\alpha, \beta]$, $\alpha \leq x \leq \beta$, where $\leq$ is defined in complete analogy to $<$, and $x \leq y \Leftrightarrow \neg(x > y)$, is also proven.
In general, if $\alpha, \beta \in \Re_{Br}$, we do not know which one is greater or equal than the other i.e., $<$ is not total upon $\Re_{Br}$. The universal law of trichotomy can be refuted, at least in an axiomatic setting of intuitionistic analysis, since

$$\neg[(\forall x, y)(x < y\ \vee\ x = y\ \vee\ y < x)]$$

can be proved[81], while for each two r.n. x, y

$$\neg\neg(x < y\ \vee\ x = y\ \vee\ y < x)$$

is also proven.
Hence, if $\alpha, \beta \in \Re_{Br}$ are given without knowing their order relation we still want to talk about their closed interval $\Delta[\alpha, \beta]$ as in classical analysis. In order to do that we define the **intuitionistic closed interval $\Delta[\alpha, \beta]$ of the arbitrarily given** $\alpha, \beta \in \Re_{Br}$ as the species of the intuitionistic reals x, such that $x > \alpha$ and $x > \beta$ is impossible and $x < \alpha$ and $x < \beta$ is also impossible. I.e., the hypotheses $x > \alpha$ and $x > \beta$ and $x < \alpha$ and $x < \beta$ lead, respectively, to an absurdity, for each element x of $\Delta[\alpha, \beta]$.

Proposition 12.10: If $\alpha, \beta \in \Re_{Br}$, then

$$\gamma = min(\alpha, \beta) = (min(q_{\alpha(n)}, q_{\beta(n)})), \quad \delta = max(\alpha, \beta) = (max(q_{\alpha(n)}, q_{\beta(n)}))$$

are also r.n.

Proof: By the definition of r.n.

$$(\forall k)(\exists n_{0\alpha})(\forall n \geq n_{0\alpha})\ |q_{\alpha(n)} - q_{\alpha(n+1)}| < \frac{1}{2^{k+1}},$$

$$(\forall k)(\exists n_{0\beta})(\forall n \geq n_{0\beta})\ |q_{\beta(n)} - q_{\beta(n+1)}| < \frac{1}{2^{k+1}}.$$

If $n_0 = max(n_{0\alpha}, n_{0\beta})$, then for each $n \geq n_0$

$$|\ q_{\gamma(n)} - q_{\gamma(n+1)}\ | = |min(q_{\alpha(n)}, q_{\beta(n)}) - min(q_{\alpha(n+1)}, q_{\beta(n+1)})| < \frac{1}{2^{k+1}},$$

[81]See [Troelstra, van Dalen 1988a] pp.257-58. CP can be used in the proof of this negation.

and

$$| q_{\delta(n)} - q_{\delta(n+1)} | = |max(q_{\alpha(n)}, q_{\beta(n)}) - max(q_{\alpha(n+1)}, q_{\beta(n+1)})| < \frac{1}{2^{k+1}}.$$

To show these inequalities it suffices to take all four cases regarding $min(q_{\alpha(n)}, q_{\beta(n)})$ and $max(q_{\alpha(n)}, q_{\beta(n)})$ and see that the second couple of inequalities is clearly justified in each case by the first one.$\diamond$

Hence, although we cannot know if $max(\alpha, \beta)$ or $min(\alpha, \beta)$ is α or β, these numbers can be defined and they behave in an expected way, as the following proposition shows:

Proposition 12.11: If $\alpha, \beta \in \Re_{Br}$, then
(i) $max(\alpha, \beta) \not< \alpha$, where $x \not< y \equiv \neg(x < y)$, $max(\alpha, \beta) \not< \beta$, $max(\alpha, \beta) = max(\beta, \alpha)$, $min(\alpha, \beta) \not> \alpha$, where $x \not> y \equiv \neg(x > y)$, $min(\alpha, \beta) \not> \beta$, $min(\alpha, \beta) = min(\beta, \alpha)$.
(ii) $x > max(\alpha, \beta) \Leftrightarrow x > \alpha \ \wedge \ x > \beta$,
$x < min(\alpha, \beta) \Leftrightarrow x < \alpha \ \wedge \ x < \beta$.
(iii) $max(\alpha, \beta) \not< min(\alpha, \beta)$.

Proof: All the above properties are trivial consequences of the definition of order.$\diamond$

A very useful property of a canonical representation of a r.n is given next.

Proposition 12.12: If $\gamma, \delta \in \Re_{Br}$ and $\delta \not< \gamma$, then there exist canonical representations $\frac{\gamma_n}{2^n}, \frac{\delta_n}{2^n}$ of γ, δ respectively such that

$$\gamma_n \leq \delta_n.$$

Proof: Equivalently, if $\alpha \not> 0$, then $\alpha_n \leq 0$, if $\alpha = (\frac{\alpha_n}{2^n})$, meaning that after some finite terms all terms are ≤ 0 and we can replace the other > 0 finite terms by 0. Similarly, if $\alpha \not< 0$, we can take all α_n to be ≥ 0. We show this last fact.
By the definition of order, $\alpha < 0 \Leftrightarrow (\exists k)(\exists n_0)(\forall n \geq n_0)\ 0 - q_{\alpha(n)} > \frac{1}{2^k} \Leftrightarrow q_{\alpha(n)} < -\frac{1}{2^k}$.
Hence,

$$\alpha \not< 0 \Leftrightarrow (\forall k)(\forall n_0)(\exists n \geq n_0)\ q_{\alpha(n)} \geq -\frac{1}{2^k} \Leftrightarrow -q_{\alpha(n)} \leq \frac{1}{2^k} \quad (*).$$

Suppose now that $\alpha_n < 0$, for each n. Then, $-\frac{\alpha_n}{2^n} > 0$, and since $\alpha_n \in \mathbb{Z}$, then $-\alpha_n \geq 1$, therefore

$$\frac{-\alpha_n}{2^n} \geq \frac{1}{2^n} \quad (**).$$

Since $(*)$ holds for each k and $(**)$ holds for each n, we get a contradiction.$\diamond$

Regarding our last argument, note that what we wanted to prove was that each canonical representation of α has only finite terms < 0, and replacing these terms by 0, we get $\alpha_n \geq 0$, for each n. I.e., we had to exclude the case of having infinite terms $\alpha_n < 0$. But such infinite terms form a (Cauchy) subsequence of an initial canonical representation of α, which is thus a canonical representation of α too. That is why we supposed $\alpha_n < 0$, for each n.
A simple consequence of the previous result is the following often used property of order.

Proposition 12.13: If x, γ, δ r.n., then
(i) $x > \delta \wedge \delta \not< \gamma \Rightarrow x > \gamma$.
(ii) $x < \delta \wedge \gamma \not< \delta \Rightarrow x < \gamma$.

Proof: (i) By Proposition 12.12, if $\delta \not< \gamma$, then there exist canonical representations $\frac{\gamma_n}{2^n}, \frac{\delta_n}{2^n}$ such that $\gamma_n \leq \delta_n$. Since there is a representation (x_n) of x, such that

$x_n > \frac{\delta_n}{2^n} \geq \frac{\gamma_n}{2^n}$ we get $x > \gamma$.
(ii) is proved similarly.$\diamond$

Next result shows why the introduction of $min(\alpha, \beta)$ and $max(\alpha, \beta)$ is important:

Proposition 12.14: If $\alpha, \beta \in \Re_{Br}$ and $\gamma = min(\alpha, \beta), \delta = max(\alpha, \beta)$, then

$$\Delta[\alpha, \beta] = \Delta[\gamma, \delta].$$

Proof: We show that the above two species are subspecies of each other, therefore equal.
(i) $\Delta[\alpha, \beta] \preceq \Delta[\gamma, \delta]$: By definition

$$x \in \Delta[\alpha, \beta] \Leftrightarrow \neg(x > \alpha \wedge x > \beta) \ \wedge \ \neg(x < \alpha \wedge x < \beta).$$

Suppose $x \in \Delta[\alpha, \beta]$. We show that

$$\neg(x > \gamma \wedge x > \delta) \ \wedge \ \neg(x < \gamma \wedge x < \delta).$$

Suppose $x > \delta$. Then, by Proposition 12.11(ii), $x > \alpha$ and $x > \beta$, which leads, by hypothesis, to a contradiction. We work similarly, if $x < \gamma$.
(ii) $\Delta[\alpha, \beta] \succeq \Delta[\gamma, \delta]$: We work exactly the same way as in case (i).$\diamond$

Proposition 12.15: If $x, \alpha, \beta \in \Re_{Br}$ and $\alpha \not> \beta$, then

$$x \in \Delta[\alpha, \beta] \Leftrightarrow x \not< \alpha \wedge x \not> \beta.$$

Proof: ($\Rightarrow$) Suppose $x \in \Delta[\alpha, \beta] \Leftrightarrow \neg(x > \alpha \wedge x > \beta) \ \wedge \ \neg(x < \alpha \wedge x < \beta)$ and $x < \alpha$. Then, since by hypothesis $\alpha \not> \beta$, Proposition 12.13(ii) gives $x < \beta$, which, together with $x < \alpha$, lead to absurdity. If $x > \beta$, then by Proposition 12.13(i), we get $x > \alpha$ and finally an absurdity again.
($\Leftarrow$) Our hypotheses are $x \not< \alpha$ and $x \not> \beta$ and $\alpha \not> \beta$. If we suppose $x > \beta$, then this contradicts $x \not> \beta$, while if we suppose $x < \alpha$, then this contradicts $x \not< \alpha$.$\diamond$

Proposition 12.16: If $x, \alpha, \beta \in \Re_{Br}$, then

$$x \in \Delta[\alpha, \beta] \Leftrightarrow x \not< \gamma \wedge x \not> \delta,$$

where $\gamma = min(\alpha, \beta), \delta = max(\alpha, \beta)$.

Proof: This is an immediate consequence of the previous proposition, since $\gamma \not> \delta$, by Proposition 12.11(iii), and $\Delta[\alpha, \beta] = \Delta[\gamma, \delta]$.$\diamond$

We shall also need the following properties of the absolute value $|\alpha|$ of a r.n. α, which does not behave as the classical one.

Proposition 12.17: If $|\alpha| = max(\alpha, -\alpha) = (|q_{\alpha(n)}|)$, then

$$|\alpha| + |\beta| \not< |\alpha + \beta|, \quad |\alpha + \beta| \not< ||\alpha| - |\beta||.$$

Proof: Suppose $|\alpha| + |\beta| \not< |\alpha + \beta|$. Then, $\exists k, \exists n_0, n \geq n_0 \ |q_{\alpha(n)} + q_{\beta(n)}| - (|q_{\alpha(n)}| + |q_{\beta(n)}|) > \frac{1}{2^k}$. But this inequality is not true in the species $\mathbb{Q}_{Br}$ which behaves as the classical set $\mathbb{Q}$. We work the same way for the other property.$\diamond$

The following theorem characterizes a closed interval $\Delta[\alpha, \beta]$ as a fan, a fact proven independently from BFT. Note that a subset of Baire space $\mathcal{N}$ is compaxt iff is the body of a fan (see Appendix, Proposition A.6).

Proposition 12.18 [Brouwer 1919]: A closed interval $\Delta[\alpha,\beta]$ of $\Re_{Br}$ is equal, as a species, to a fan $\mathcal{F}_{[\alpha,\beta]}$.

Proof: If we define $\gamma = min(\alpha,\beta)$, $\delta = max(\alpha,\beta)$, then, by Proposition 12.12, $\Delta[\alpha,\beta] = \Delta[\gamma,\delta]$. Also, by Proposition 12.12 we can represent γ,δ in a canonical way by sequences $\frac{\gamma_n}{2^n}, \frac{\delta_n}{2^n}$, where $\gamma_n, \delta_n \in \mathbb{Z}$, and $\gamma_n \leq \delta_n$.
Let $\mathcal{F}_{[\alpha,\beta]}$ be the spread of c.r.n. $(\frac{x_n}{2^n})$, satisfying

$$\gamma_n \leq x_n \leq \delta_n,$$

for each n. We show that $\mathcal{F}_{[\alpha,\beta]}$ is actually a fan. By definition

$$|\frac{x_n}{2^n} - \frac{x_{n+1}}{2^{n+1}}| < \frac{1}{2^n},$$

for each n, therefore

$$\frac{x_n}{2^n} - \frac{1}{2^n} < \frac{x_{n+1}}{2^{n+1}} < \frac{x_n}{2^n} + \frac{1}{2^n} \Leftrightarrow$$
$$2x_n - 2 < x_{n+1} < 2x_n + 2.$$

Since x_n, x_{n+1} are integers, x_{n+1} can take only the values: $2x_n - 1, 2x_n, 2x_n + 1$ and consequently, the possible immediate successors of $\frac{x_n}{2^n}$ are

$$\frac{2x_n - 1}{2^{n+1}} = \frac{x_n}{2^n} - \frac{1}{2^{n+1}}, \quad \frac{2x_n}{2^{n+1}} = \frac{x_n}{2^n}, \quad \frac{2x_n + 1}{2^{n+1}} = \frac{x_n}{2^n} + \frac{1}{2^{n+1}}$$

respectively. Therefore, $\mathcal{F}_{[\alpha,\beta]}$ is a finitely branching spread. Now, by Proposition 12.14 it suffices to show that

$$\mathcal{F}_{[\alpha,\beta]} = \Delta[\gamma,\delta].$$

$\mathcal{F}_{[\alpha,\beta]} \preceq \Delta[\gamma,\delta]$: We apply Proposition 12.16. Suppose $x \in \mathcal{F}_{[\alpha,\beta]}$ and $x > \delta$. Then, $\frac{x_n}{2^n} > \frac{\delta_n}{2^n}$ i.e., $x_n > \delta_n$, which contradicts the definition of $\mathcal{F}_{[\alpha,\beta]}$. Similarly we reach a contradiction supposing $x < \gamma$. Hence $x \in \Delta[\gamma,\delta]$.
$\mathcal{F}_{[\alpha,\beta]} \succeq \Delta[\gamma,\delta]$: If $x \in \Delta[\gamma,\delta]$, then, by Proposition 12.16, $x \not< \gamma \wedge x \not> \delta$. Since $x \not< \gamma$, then by Proposition 12.12, $\gamma_n \leq x_n$ and since $x \not> \delta$, then $x_n \leq \delta_n$ and consequently $\gamma_n \leq x_n \leq \delta_n$ i.e., $x \in \mathcal{F}_{[\alpha,\beta]}$.$\diamond$

Now we are ready to prove, using BFT, Brouwer's Uniform Continuity theorem.

Proposition 12.19 Brouwer's Uniform Continuity theorem (UCT 1923): If $\Phi : \Delta[\alpha,\beta] \to \Re_{Br}$ is an intuitionistic Function, then Φ is uniformly continuous.

Proof: By the previous proposition, Φ is actually a Function

$$\Phi : \mathcal{F}_{[\alpha,\beta]} \to \Re_{Br},$$

and

$$x \mapsto \Phi(x) = \xi = (\frac{\xi_n}{2^n}).$$

For each $\nu \in \omega$ we define an intuitionistic function

$$\varphi_\nu : \mathcal{F}_{[\alpha,\beta]} \to \omega,$$

by

$$\varphi_\nu(x) = \xi_\nu.$$

Since $\mathcal{F}_{[\alpha,\beta]}$ is a fan, BFT ensures the existence of a natural number $N = N(\nu)$ such that

$$\forall x \forall y\ N_x = N_y \Rightarrow \varphi_\nu(x) = \varphi_\nu(y).$$

Then we show that this natural number N satisfies the following condition

$$|x-y| < \frac{1}{2^{N+1}} \Rightarrow |\Phi(x) - \Phi(y)| < \frac{5}{4}\frac{1}{2^\nu},$$

which guarantees, since it holds for each ν, that Φ is uniformly continuous.
If $|x-y| < \frac{1}{2^{N+1}}$ we can consider, without loss of generality, that the canonical representations $(\frac{x_n}{2^n}), (\frac{y_n}{2^n})$ satisfy

$$x_1 = y_1, \quad x_2 = y_2, \ \ldots, \ x_N = y_N.$$

To see this we write

$$0 < |x-y| = (|\frac{x_n}{2^n} - \frac{y_n}{2^n}|) = (\frac{|x_n - y_n|}{2^n}) < \frac{1}{2^{N+1}}$$

and since $(\frac{|x_n - y_n|}{2^n})$ is clearly a canonical representation of $|x-y|$ ($(\frac{x_n - y_n}{2^n})$ is a canonical representation of $x-y$), we may choose

$$0 \leq \frac{|x_n - y_n|}{2^n} \leq \lambda_n,$$

where λ_n is the following canonical representation of $\frac{1}{2^{N+1}}$:

$$\lambda_n = \begin{cases} \frac{0}{2^n} & \text{, if } n < N+1 \\ \frac{2^{n-(N+1)}}{2^n} = \frac{1}{2^{N+1}} & \text{, if } n \geq N+1 \end{cases}$$

Hence, the first N terms of $\frac{|x_n - y_n|}{2^n}$ are 0 and consequently $x_1 = y_1, \ x_2 = y_2, \ \ldots\ x_N = y_N$.
Thus, sequences $(\frac{x_n}{2^n}), (\frac{y_n}{2^n})$ have the same initial N-segment.
Therefore, for $\Phi(x) = \xi = (\frac{\xi_n}{2^n})$ and $\Phi(y) = \zeta = (\frac{\zeta_n}{2^n})$

$$N_x = N_y \Rightarrow \xi_\nu = \zeta_\nu = \eta.$$

By (3) of Proposition 12.9,

$$\frac{5}{8}\frac{1}{2^\nu} > |\Phi(x) - \frac{\eta}{2^\nu}|, \quad \frac{5}{8}\frac{1}{2^\nu} > |\Phi(x) - \frac{\eta}{2^\nu}|$$

hence[82],

$$\frac{5}{4}\frac{1}{2^\nu} > |\Phi(x) - \frac{\eta}{2^\nu}| + |\Phi(x) - \frac{\eta}{2^\nu}| \not< |\Phi(x) - \Phi(y),$$

therefore, by Proposition 12.13(i),

$$\frac{5}{4}\frac{1}{2^\nu} > |\Phi(x) - \Phi(y)|,$$

[82]Using the obvious property $x > \alpha, y > \beta \Rightarrow x + y > \alpha + \beta$.

and UCT is established.$\diamond$

Of course, UCT is classically false. E.g., $f : [-1,1] \to \mathbb{R}$ with

$$f(x) = \begin{cases} 0 & , \text{ if } x \leq 0 \\ 1 & , \text{ if } x > 0 \end{cases}$$

is discontinuous at 0. Intuitionistically though, the order $<$ of $\Re_{Br}$ does not satisfy[83]. the classically true disjunction

$$(\forall x)(x \leq 0 \ \vee \ x > 0),$$

therefore, the above Function f is not totally defined on $[-1,1]$. I.e., the fact that an intuitionistic Function is totally defined on an interval is a much stronger hypothesis which entails its uniform behavior.

And what about discontinuous functions within intuitionism? The answer is given in Weyl's 1920 Zürich lectures and in Brouwer's approving editorial notes[84].

> [**(Weyl)** "It is clear that one cannot explain the concept 'continuous function in a bounded interval' without including 'uniform continuity' and 'boundedness' in the definition. Above all, there cannot be any function in a continuum other than continuous functions. When the Old Analysis introduced 'discontinuous functions' it showed most clearly how far it had departed from a clear understanding of the essence of the continuum. What is nowadays called a discontinuous function is in reality no more than a number of functions on separate continua..."
> **(Brouwer)** "Better to say 'the function is not everywhere defined'."
> **(Weyl)** "Take for example the continua $C, C^+(x > 0)$ and $C^-(x < 0)$... If we consider the two functions $+1$ in C^+ and -1 in C^- then *there does not exist* a function defined on the whole of C equivalent with the one value for C^+ and the other value for C^-."
> **(Brouwer)** "Very true! Underline, because this is the main and most important point."]

Proposition 12.20: If $f : \Re_{Br} \to \omega$, is an intuitionistic $\Re_{Br}$-function, then f is constant.

Proof: A $\Re_{Br}$-function $f : \Re_{Br} \to \omega$ can be seen as a Function $F : \Re_{Br} \to \omega^\omega$, where

$$F(\alpha) = \overline{n},$$

and $\overline{n}$ is the constant sequence $(n, n, n, ...)$.

If α, β two different real numbers, the interval $[\alpha, \beta]$ is a fan and $F_{|[\alpha,\beta]}$ satisfies UCT i.e., if $1 > \varepsilon > 0$,

$$(\exists\delta)(\forall\gamma)(\forall\beta)(|\gamma - \beta| < \delta \Rightarrow |F(\gamma) - F(\delta)| < \varepsilon),$$

which becomes

$$(\exists\delta)(\forall\gamma)(\forall\beta)(|\gamma - \beta| < \delta \Rightarrow F(\gamma) = F(\delta)).$$

[83] See e.g., [Truss 1997] p.319.

[84] We quote from [van Stigt 1990] p.379.

Since we can cover $[\alpha, \beta]$ by a finite number of intervals of length δ, then, starting from α, we conclude that $F(\alpha) = F(\beta)$.$\diamond$

If T is a subspecies of S, then T is called a *detachable subspecies* of S iff

$$S = T \vee (S - T).$$

Proposition 12.21: If T is a detachable subspecies of $\Delta[\alpha, \beta]$, then T is the empty species or the whole of $\Delta[\alpha, \beta]$.

Proof: Since T is a detachable subspecies of $\Delta[\alpha, \beta]$, then the following function

$$f_T(x) = \begin{cases} 1 & \text{, if } x \in T \\ 0 & \text{, if } x \notin T \end{cases}$$

is totally defined on $\Delta[\alpha, \beta]$, therefore, arguing as in the proof of Proposition 12.20, f_T is constant. If $f_T = 1$, then $T = \Delta[\alpha, \beta]$, while if $f_T = 0$, then T is the empty species.$\diamond$

The full development of $\Re_{Br}$ is beyond the scope of our Thesis. What we presented here is only the foundational core of Brouwer's Intuitionistic Analysis. Many theorems of classical analysis, like the Bolzano-Weierstrass theorem, or the intermediate value theorem fail, but intuitionistic counterparts of them are studied (see e.g., [Heyting 1966] and [Troelstra, van Dalen 1988a]).

13. Post-Brouwer formulations of Fan theorem. We present some results around fan theorem and some generalizations of BFT connecting BIA to post-Brouwer intuitionistic analysis. These concepts and results form part of the way Brouwer's intuitionism was received and presented in post-Brouwer era.
The necessity of bar's decidability: As we have already said, Brouwer didn't mention explicitly in the hypotheses of his Bar theorem the decidability of bar B. Kleene, in [Kleene, Vesley 1965] pp.87-88, showed that the decidability of B is necessary. We present here a simplified version of Kleene's counterexample given by van Dalen in [Brouwer 1981] p.102):

Proposition 13.1 (Kleene's counterexample (KC)): There exists a spread M and an undecidable bar B of M such that B has no thin decidable sub-bar B_0, therefore Bar theorem fails for spread M and its bar B.

Proof: Let ω^ω be the universal spread and A a decidable predicate on the naturals i.e.,

$$\forall n(A(n) \vee \neg A(n)),$$

such that the following disjunction doesn't hold:

$$\forall n A(n) \vee \neg[\forall n A(n)]$$

i.e., we do not posses yet a proof neither of $\forall n A(n)$, nor of $\neg[\forall n A(n)]$. E.g, we may consider the predicate $A(n)$ iff $2n$ is sum of two odds (Goldbach's conjecture).
We then define B as follows:
$<> \in B \Leftrightarrow \neg[\forall n A(n)]$,
$(n) \in B \Leftrightarrow A(n)$,
where (n) is a sequence of length 1, while it is irrelevant which other finite sequences

belong to B. Then:
(i) B **is a bar of** ω^ω**:** Let $\alpha \in \omega^\omega$. By the decidability of $A(n)$, $A(\alpha(1)) \vee \neg A(\alpha(1))$. If $A(\alpha(1))$, then $\alpha(1) \in B$, hence α cuts B, while if $\neg A(\alpha(1))$, there is a counterexample to $\forall n A(n)$, thus $<> \in B$ i.e., α cuts B at the root $<>$.
(ii) B **is not decidable:** Suppose that it was. Then, $<>$ would satisfy

$$(1) \qquad <> \in B \vee <> \notin B,$$

which by the definition of B amounts to:

$$(2) \qquad \neg[\forall n A(n)] \ \vee \ \neg\neg[\forall n A(n)].$$

Intuitionistically the following is proved:

$$(3) \qquad \neg\neg[\forall n A(n)] \Rightarrow \forall n \neg\neg A(n),$$

while by the decidability of $A(n)$ we get:[85]

$$(4) \qquad \forall n \neg\neg A(n) \Rightarrow \forall n A(n).$$

By (2), (3) and (4) we get $\forall n A(n) \vee \neg[\forall n A(n)]$, which by hypothesis is not true on A. Therefore, B cannot be decidable.
(iii) B **has no thin decidable sub-bar:** If there existed such sub-bar B_0, then

$$<> \in B_0 \vee <> \notin B_0,$$

would hold for B_0. But then, with an argument similar to the one used in (ii), we reach an absurdity about our knowledge of A.$\diamond$

Thus, we showed that BT is intuitionistically unacceptable without decidability condition of bar B. Of course, classically it is still true, since decidability is trivially true, while intuitionistically is unacceptable, since there are predicates like A. Actually, the above counterexample is a weak counterexample. Although decidability of a φ-bar is obvious, Brouwer's proof of BT in [Brouwer 1981]) doesn't mention B's decidability, therefore, by the previous proposition, it is false.
Kleene's counterexample is quite expected, since in order to find a bound on the length of sequences of a fan we have to be able to recognize when a sequence cuts the bar or not. Brouwer never mentioned explicitly bar's decidability, since the φ-bar he used is always decidable. The above result though, is in accordance to the dependence, in the proof of BT, of B_0's decidability on the decidability of B.

As we have already said in Paragraph 2, König's lemma, that an infinite classical fan has an infinite branch, has a highly non-constructive classical proof. A branch of a classical fan may have terminating branches, while an intuitionistic fan has non-terminating branches[86]. The following proposition shows why König's lemma is intuitionistically

[85]In (3) hypothesis $\neg\neg[\forall n A(n)]$ means $\neg[\forall n A(n)] \Rightarrow \perp$. If we fix n, we show that $\neg\neg A(n)$ i.e., $\neg A(n) \Rightarrow \perp$. But, if $\neg A(n)$, then $\neg[\forall n A(n)]$, thus, by hypothesis we reach $\perp$.
Regarding (4), it suffices to show that for each n, $A(n)$. By the hypothesis of (4) though on that n, we get $\neg\neg A(n)$, hence, by the decidability hypothesis on A we get $A(n)$.

[86]As we have noted in Paragraph 5, although this difference has prevailed, Brouwer in some of his formulations of the definition of a spread permitted terminating branches.

unacceptable.

Proposition 13.2: There is a classical fan T for which König's lemma is intuitionistically unacceptable.

Proof: Let T be the classical fan defined as the following tree: After root $<>$ only the constant sequences $\overline{0}$ and $\overline{1}$ start, one of which may terminate, without knowing though, which one and when that may happen. This is achieved through an unsolved problem, or through the will of the creating subject. E.g., let A be a decidable predicate on natural numbers, such that we do not know a proof neither of $\forall nA(n)$ nor of $\neg[\forall nA(n)]$. Suppose that if we find an odd n such that $\neg A(n)$, then $\overline{1}$ terminates, otherwise, if we find a proof of $\forall nA(n)$, it is $\overline{0}$ that terminates. Both continue until the knowledge of $\forall nA(n)$ or $\neg A(n)$, for some n, terminates one of them.
Obviously, T is a (classical infinite) fan, since if we suppose T is finite, then both branches terminate, which is absurd, by the definition of Λ_T. Intuitionistically speaking though, T has no infinite branch, since we cannot know which of the two branches is the non-terminating one. To know that presupposes that we know the answer to the initial up till now unsolved problem about $A(n)$. Classically, its solution is independent from our knowledge of it, therefore one of the two branches is (logically) infinite, while intuitionistically it is only impossible that both branches are terminating ones. To say though, that one of them is infinite presupposes that we have determined exactly which is the infinite one, something which is impossible.$\diamond$

As we have already suggested, different versions of fan theorem have prevailed in post-Brouwer intuitionistic literature. Moreover, these formulation of fan theorem are classically equivalent to König's lemma, while, by the previous proposition that could be intuitionistically rather "annoying". The main reason for that was Kleene's system FIM of intuitionistic analysis in which **Kleene's fan theorem (KFT)** has the following formulation[87]:

$$(KFT) \qquad (\forall\alpha \in 2^\omega)\exists nR(\alpha, n) \Rightarrow$$
$$(\exists m)\ (\forall\alpha \in 2^\omega)\ (\exists n)\ (\forall\gamma \in 2^\omega)\ [m_\alpha = m_\gamma \Rightarrow R(\gamma, n)]$$

where $R(\alpha, n)$ a relation between sequences and natural numbers.
Another form of fan theorem is found e.g., in [Beeson 1985] p.50, which we call **monotone fan theorem (MFT)**:

(MFT): Let R be a predicate on finite $0, 1$-sequences such that:
(i) $[(u \prec v) \wedge R(u)] \Rightarrow R(v)$ i.e., R is a monotone predicate.
(ii) $(\forall\alpha \in 2^\omega)(\exists n)R(n_\alpha)$
Then,

$$(\exists m)\ (\forall\alpha \in 2^\omega)\ R(m_\alpha)$$

i.e., all finite sequences of length m satisfy predicate R.
Another formulation of fan theorem is found e.g., in [Troelstra, van Dalen 1988a]), which we call **decidable fan theorem (DFT)**:

(DFT): Let $A(\alpha)$ be an intuitionistic predicate on a fan T (not necessarily monotone on the nodes of T) such that:

[87] As we show in Proposition 8.1.7, it suffices to formulate fan theorem on the special fan 2^ω.

(i) A is decidable i.e.,

$$A(\alpha_1, \alpha_2, ..., \alpha_n) \vee \neg A(\alpha_1, \alpha_2, ..., \alpha_n),$$

for each n and for each α in $[T]$.
(ii) $(\forall \alpha \in [T])(\exists n)A(n_\alpha)$
Then,

$$(\exists N)(\forall \alpha \in [T])(\exists m \leq N)A(m_\alpha).$$

I.e., all finite sequences of T satisfy predicate A for the first time before their length becomes $N+1$. Classically, the decidability condition $A(\alpha_1, \alpha_2, ..., \alpha_n) \vee \neg A(\alpha_1, \alpha_2, ..., \alpha_n)$ holds in a trivial way.

The following propositions concern the relation between the above formulations of fan theorem.

Proposition 13.3: The following are only classically equivalent i.e., equivalent using principles of classical logic, which are not accepted in intuitionistic logic.
(i) KL.
(ii) DFT.

Proof: We use KL in the form KL_2, according to which, if all branches of a fan T are finite, then T has a branch of maximum length.
$(i) \Rightarrow (ii)$ Through fan T and predicate A we define the tree

$$T_A = \{(\alpha_0, \alpha_1, ..., \alpha_m) \in T \ : \forall k \leq m \ \ \neg A(\alpha_0, \alpha_1, ..., \alpha_k)\}$$

i.e., T_A contains all elements of T all the initial segments of which are not A-satisfiable. T_A is a fan, since it is a sub-tree of the fan T, and each branch of T_A is finite, since, by the hypothesis on A, each sequence of T will satisfy A at some moment. Therefore, T_A satisfies the hypothesis of KL_2.
Suppose that T violates the conclusion of DFT i.e.,

$$\neg[(\exists N)(\forall \alpha \in [T])(\exists m \leq N)A(m_\alpha)] \Leftrightarrow [(\forall N)(\exists \alpha)(\forall m \leq N)\neg A(m_\alpha)].$$

Thus, for each N there is a branch of T which belongs to T_A, since for each $m \leq N$, $\neg A(\alpha_0, \alpha_1, ..., \alpha_m)$. Hence, T_A has not a branch of maximum length, something which violates KL_2. So, we have shown that it is impossible the conclusion of DFT to be violated, which classically means that the conclusion of DFT actually holds.

$(ii) \Rightarrow (i)$ Suppose T is a fan all the branches of which are finite. A branch of T is called *maximal* iff it is not extended. We define the following predicate A on T:

$$A(\alpha_0, \alpha_1, ..., \alpha_m) \text{ iff } (\alpha_0, \alpha_1, ..., \alpha_m) \text{ is a maximal branch of } T.$$

We extend the finite nodes of T adding constantly a symbol $*$ such that a new tree T^* is formed the infinite branches of which are of the form

$$(\alpha_0, \alpha_1, ..., \alpha_m, *, *, *, ...),$$

where $(\alpha_0, \alpha_1, ..., \alpha_m)$ is a maximal branch of T. T^* is a fan and A is defined on T^*'s nodes too: if u is node of T^*, then $A(u)$ iff u is T^*-maximal i.e., $v \succ u \Rightarrow v$ contains

the $*$ symbol.
Since $(\forall\alpha \in [T^*])(\exists n)A(n_\alpha)$, the conclusion of DFT guarantees that there is a branch of T of maximum length.$\diamond$

Since DFT is classically equivalent to intuitionistically unacceptable KL, it is even more suitable that BIA includes BFT, which is not classically valid. As the following proposition shows though, DFT "entails" BFT, given an intuitionistic function φ. In the literature we find it in the form

$$DFT + CP \Rightarrow BFT,$$

since it is not a definition of an intuitionistic function that founds this concept, but CP as an axiom. In our framework though, the starting point of any conclusion regarding intuitionistic functions is their definition.

Proposition 13.4: If $\varphi : T \to \omega$ is an intuitionistic function on a fan T, then, if DFT is typically applied, BFT is derived.

Proof: By the definition of a function $\varphi : T \to \omega$ the value $\varphi(\alpha)$ of a sequence α in T is determined by some initial segment n_α of α. Let A be the following predicate on the nodes of T:

$$A(n_\alpha) \text{ iff } n_\alpha \text{ is critical to } \varphi.$$

A is an intuitionistic predicate and a monotone one, since an extension in T of a critical to φ^* node is also critical. Also, $(\forall\alpha \in T)\exists n A(n_\alpha)$, since φ is fully defined on T and φ is an intuitionistic function. Clearly, the conclusion of DFT gives the conclusion of BFT, since predicate A is monotone.$\diamond$

We see that we have to add the non-classical concept of intuitionistic function to the classically accepted DFT in order to get the non-classical BFT. I.e., DFT is a classical proposition which entails BFT, only if we limit the concept of classical function to that of intuitionistic one.

Kleene's system FIM of intuitionistic analysis includes the following two principles-axioms:

(i) **Kleene's Continuity Principle (KCP):**

$$(KCP) \quad (\forall\alpha \in 2^\omega)\ \exists n R(\alpha, n) \Rightarrow [\forall\alpha\ \exists m, n\ \forall\gamma\ (m_\alpha = m_\gamma \Rightarrow R(\gamma, n))].$$

(ii) **Decidable Bar Induction (DBI):** If M is a spread and
(I) $(\forall\alpha \in M)\ (\exists n)(n_\alpha \in B)\ \wedge$
(II) $(\forall u \in M)\ (u \in B \vee u \notin B)\ \wedge$
(III) $(\forall u \in M)\ (u \in B \Rightarrow u \in W)\ \wedge$
(IV) $(\forall u \in M)\ (\forall k(u \frown k \in W) \Rightarrow u \in W)$,
Then, $\quad <> \in W$,
where $u \frown k$ is an immediate successor of node u[88].

Among the above axioms of FIM, KCP is the only non-classical principle of FIM, while DBI, which codifies Brouwer's proof of BFT avoiding Brouwer's dogma, is classically

[88]We use again the same symbol $\alpha \in M$ and $u \in M$ for infinite and finite M-sequences.

true.
Interpreting the hypotheses of DBI we get:
(I) $(\forall \alpha \in M)$ $(\exists n)(n_\alpha \in B)$: the species B is a bar of spread M.
(II) $(\forall u \in M)$ $(u \in B \vee u \notin B)$: bar B is decidable.
(III) $(\forall u \in M)$ $(u \in B \Rightarrow u \in W)$: B is a subspecies of a species W of finite - sequences.
(IV) $(\forall u \in M)$ $(\forall k(u \frown k \in W) \Rightarrow u \in W)$: if all immediate M-successors of an M-node u satisfy property W, then W is inherited to u.
The conclusion of DBI expresses the fact that W is inherited to the root $<>$ of spread M. Note that species W need not be decidable.

DBI is an inverse kind of induction. While in traditional induction on naturals there is a fixed initial point 0 or 1, in DBI there is a fixed final point, the root $<>$. The basis of DBI is condition $(\forall \alpha \in M)\exists n(n_\alpha \in B)$. The constructive content of DBI is discussed in Paragraph 15. The general idea of DBI is that starting from the elements of B which satisfy property W, we "go down" through $(\forall u \in M)$ $(\forall k(u \frown k \in W) \Rightarrow u \in W)$, and finally we reach $<>$ satisfying W too. This route is a generalization of the root of a proof R_u in Brouwer's proof of BFT. The intuitionistic legitimacy of post-Brouwer principle DBI is seemingly justified by the following proposition.

Proposition 13.5: DBI entails Brouwer's Bar theorem BBT, which in turn entails BFT (Proposition 11.7).

Proof: Defining species W through $u \in W$ iff u has the well-ordering property for nodes, we get the conclusion of BBT i.e., that the root $<>$ has the well-ordering property too.$\diamond$

As we have already mentioned, DBI holds classically. After the proof of this fact we discuss it from the intuitionistic point of view.

Proposition 13.6: DBI holds classically.

Proof: Let properties B and W, satisfying hypotheses of DBI. Suppose that $<> \notin W$. Hence, there is a natural number α_1 such that the 1-sequence (α_1) is M-accepted and $(\alpha_1) \notin W$. Moreover, we may consider α_1 as the minimum natural with that property. Sequence (α_1) has for the same reason an immediate successor node (α_1, α_2) which does not satisfy W either. In that way there exists classically a sequence, each initial segment of which is not in W, which is absurd, since this sequence cuts B at some point , therefore it has a node in W. Hence, $<> \in W$.$\diamond$

Obviously, the above proof uses the same choice principle and the same philosophy of mathematical existence found in the proof of König's lemma. Also, we find in it the same contraposition with the one used in the classical proof of fan theorem through König's lemma. Actually, Dummett, in [Dummett 2000] p.56, considers this fact as an encouraging element fo the intuitionistic validity of DBI.
Hence, classically, given the hypotheses of DBI, the following implication holds:

$$<> \notin W \Rightarrow \exists k \ (k) \notin W,$$

where (k) M-accepted 1-sequence. Intuitionistically though, only

$$<> \notin W \Rightarrow \neg(\forall k \ (k) \in W)$$

holds. In order to accept the first implication, Markov's principle and decidability of W are needed. We just note here that Brouwer didn't accept Markov's principle and the decidability of W is not in the hypotheses of DBI. Even if though, the decidability of W has to be accepted, since at some moment for the decidable W a natural k must occur for which $(k) \notin W$ (otherwise $\forall k\ (k) \in W$ would hold), one could conclude only

$$\neg(<> \notin W),$$

since an on-going sequence is constructed, no initial segment of which belongs to B. But what was needed was to conclude $<> \in W$.
Even if we limit to a fan, and then only a finite number of immediate successors needs to be checked each time and Markov's principle is avoided, we only conclude again that $\neg(<> \notin W)$, with W being decidable. If W is not decidable and there are two immediate successor nodes of u for which we do not know if they belong to W or not, even if we know that $\neg(\forall k\ u \frown k \in W)$, we cannot conclude that $\exists k\ u \frown k \notin W$, since we cannot say which one of the two successor nodes is the one in W.
Someone could say that adding the decidability of W one is very close, if the spread is a fan, to the intuitionistic proof of DBI. Lack of decidability though is essential. Moreover, property W of Brouwer's proof of BFT, that node u has the well-ordering property of nodes, is *not* decidable.

We examine now if the following conjunction is accepted:

$$(*) \quad <> \in W \ \wedge\ \exists k\ (k) \notin W.$$

If we accept Markov's principle, or if we study a fan with a decidable W, then with the argument of the previous classical proof, a potentially infinite sequence is constructed which has as 1-segment (k) and none initial segment of which is in W. Hence, $<> \in B$. If B was defined such that $<> \notin W$, we reach an absurdity.
Thus we see, that when W and B satisfy healthy conditions and we work in a fan, then the conclusion of DBI guarantees that all nodes "under" the nodes of B belong to W. Of course, $(*)$ doesn't hold generally, since if $B = W = \{<>\}$, the hypotheses of DBI are trivially satisfied, while no node (k) belongs to W.

Proposition 13.7: Assuming KCP, the following are equivalent:
(i) Kleene's fan theorem KFT.
(ii) $DFT(2^\omega)$.

Proof: $(i) \Rightarrow (ii)$ Let A be a decidable intuitionistic predicate such that $(\forall \alpha \in 2^\omega)(\exists n)A(n_\alpha)$. We then define the following relation between α, n:

$$\Sigma(\alpha, n) \Leftrightarrow\ n \text{ is the least natural: } A(n_\alpha).$$

$\Sigma(\alpha, n)$ is well-defined, since A is decidable and $(\forall \alpha \in 2^\omega)(\exists n)A(n_\alpha)$. By the conclusion of KFT we get that

$$(\exists m)\ (\forall \alpha \in 2^\omega)\ (\exists n)\ (\forall \gamma \in 2^\omega)\ [m_\alpha = m_\gamma \Rightarrow \Sigma(\gamma, n)].$$

Hence, all m-equal to α sequences have their n-initial segment as the least initial segment satisfying A. If $n \le m$, or even if $n > m$, n is a bound for all m-equal to α sequences. Since sequences of length m are finite, setting N the maximum

value of bounds n for each sequence of length m, N is the global bound satisfying $(\exists N)\ (\forall\alpha \in 2^{\omega})(\exists m \leq N)A(m_{\alpha})$.
$(ii) \Rightarrow (i)$ Suppose $(\forall\alpha \in 2^{\omega})\exists n R(\alpha, n)$. We define predicate $S(m_{\alpha})$ according to

$$S(m_{\alpha}) \Leftrightarrow (\exists n)\ (\forall\gamma \in 2^{\omega})\ [m_{\alpha} = m_{\gamma} \Rightarrow R(\gamma, n)].$$

Then, $(\forall\alpha \in 2^{\omega})(\exists m)S(m_{\alpha})$, by KCP. The conclusion of $DFT(2^{\omega})$ for S gives the conclusion of KFT for $R(\alpha, n)$, since if α, γ share the same N-initial segment, where N the global bound determined by DFT, then α, γ share the same m-initial segment, where $m \leq N$. That gives $R(\gamma, n)$.$\diamond$

KCP was used only in the $(ii) \Rightarrow (i)$ direction, therefore, KFT is more general, on its own, than DFT.

Another variation of DBI found in the literature is the **monotone bar induction principle (MBI)**, where the decidability of bar B in DBI is replaced by its monotonicity. Namely, if M is a spread and
(I) $(\forall\alpha \in M)(\exists n)(n_{\alpha} \in B)\ \wedge$
(II) $(\forall u \in M)\ (\forall v \in M)\ (u \preceq v)\ (u \in B \Rightarrow v \in B)\ \wedge$
(III) $(\forall u \in M)\ (u \in B \Rightarrow u \in W)\ \wedge$
(IV) $(\forall u \in M)\ (\forall k(u \frown k \in W) \Rightarrow u \in W)$,
Then, $<> \in W$,
MBI also holds classically, repeating the argument in the proof of Proposition 13.6. The two induction principles are equivalent.

Proposition 13.8: MBI entails DBI.

Proof: Supposing (I)-(IV) of DBI and assuming MBI i.e., monotonicity, we reach the conclusion of DBI.
We define two new predicates B' and W' from B and W as follows:

$$u \in B' \Leftrightarrow \exists v, v \preceq u : v \in B,$$
$$u \in W' \Leftrightarrow u \in W \vee u \in B'.$$

Obviously, $u \in B \Rightarrow u \in B'$. We show that B' and W' satisfy the hypotheses of MBI.
$(\forall\alpha \in M)\ \exists n(n_{\alpha} \in B')$: By condition (I) of DBI, $\exists n(n_{\alpha} \in B)$, but then $n_{\alpha} \in B'$ too.
$(\forall u \in M)\ (\forall v \in M)\ (u \preceq v)\ (u \in B' \Rightarrow v \in B')$: Since $u \in B'$, some ancestor node of u belongs to B, which is though an ancestor of v too.
B' is the first monotone property one thinks to define from B.
$(\forall u \in M)\ (u \in B' \Rightarrow u \in W')$: It holds trivially by the definition of W'.
$(\forall u \in M)\ (\forall k(u \frown k \in W') \Rightarrow u \in W')$: We first note that B' is decidable, since B is decidable, by checking the ancestors of a node with respect to B.
Let $u \in B'$. Then, directly, $u \in W'$.
Let $u \notin B'$ and suppose $u \frown k \in W'$, for each k. By the definition of W' though,

$$(1) \quad u \frown k \in W' \Rightarrow [u \frown k \in W \vee u \frown k \in B'].$$

Also,

$$(2) \quad u \frown k \in B' \Rightarrow u \frown k \in B,$$

otherwise, if $u \frown k \notin B$, the ancestor of $u \frown k$ in B is $\preceq u$, hence $u \in B'$, which is absurd, since, by hypothesis, $u \notin B'$. (III) of DBI gives

$$(3) \quad u \frown k \in B' \Rightarrow u \frown k \in W.$$

Because of (3), (1) becomes

$$(4) \quad u \frown k \in W' \Rightarrow u \frown k \in W.$$

Applying condition (IV) of DBI on (4), we get $u \in W$, which gives, by the definition of W', $u \in W'$.
Thus, B' and W' satisfy the hypotheses of MBI, and hence $<> \in W'$. But then, $<> \in W$, or $<> \in B'$. Since, $<> \in B' \Rightarrow <> \in B$, we also get $<> \in W$.$\diamond$

For a partial inverse see Proposition 13.11.
The above definition of property W' is not the only choice we have, as we show in next proposition.

The **induction principle of a classical bar (CBI)** is DBI without the decidability condition on B, which is classically trivially true. I.e., CBI is the following induction scheme. If M is a spread and
(I) $(\forall \alpha \in M)(\exists n)(n_\alpha \in B) \ \wedge$
(II) $(\forall u \in M)\ (u \in B \Rightarrow u \in W) \ \wedge$
(III) $(\forall u \in M)\ (\forall k (u \frown k \in W) \Rightarrow u \in W)$,
Then, $\quad <> \in W$,
Classically, DBI is equivalent to CBI. It is interesting to see that defining a new W' we get classically CBI from MBI, but not intuitionistically (Proposition 13.12).

Proposition 13.9: Classically, MBI entails CBI.

Proof: Following the previous line of proof, B' is the same, while W' is defined by

$$u \in W' \Leftrightarrow \exists v, v \preceq u : v \in W.$$

Obviously, $u \in W \Rightarrow u \in W'$. We show that B' and W' satisfy the hypotheses of MBI.
Hypotheses $(\forall \alpha \in M)(\exists n)(n_\alpha \in B')$ and $(\forall u \in M)\ (\forall v \in M)\ (u \preceq v)\ (u \in B' \Rightarrow v \in B')$ are proved the same way.
$(\forall u \in M)\ (u \in B' \Rightarrow u \in W')$: If $u \in B'$, then, by definition of B', $\exists v, v \preceq u : v \in B$ and by CBI, $v \in W$, therefore $u \in W'$, by the definition of W'.
$(\forall u \in M)\ (\forall k(u \frown k \in W') \Rightarrow u \in W')$: Let $u \frown k \in W'$. By the definition of W' $\exists v, v \preceq u \frown k : v \in W$. If there is k, such that $v \preceq u \frown k$ and $v \in W$, then, by definition of W', $u \in W'$. If for all k, $u \frown k \in W$, then, by the last hypothesis of CBI, we get $u \in W$, hence $u \in W'$.
Thus, by the conclusion of MBI, $<> \in W'$, therefore, $<> \in W$, since root $<>$ is the only initial node of itself.$\diamond$

The non-intuitionistic step of the above proof is the disjunction

$$(\forall k\ u \frown k \in W)\ \vee\ (\exists k\ u \frown k \notin W).$$

As it is expected, Kleene's counterexample suggests that CBI is not intuitionistically accepted.

Proposition 13.10: CBI cannot be intuitionistically accepted.

Proof: Let M be ω^ω and A Kleene's decidable predicate, for which $\forall n A(n) \vee \neg[\forall n A(n)]$.
Again we define B as follows:
$<> \in B \Leftrightarrow \neg[\forall n A(n)]$,

$(n) \in B \Leftrightarrow A(n)$, where (n) an 1-sequence.
Each extension of an 1-sequence (n), for which $A(n)$ holds, belongs to B.
We also define W by

$$u \in W \Leftrightarrow \exists v, v \preceq u : v \in B.$$

Since, as it has been shown in Kleene's counterexample, B is a bar, then $(\forall \alpha \in \omega^\omega)(\exists n)(n_\alpha \in B)$. Also, B is not decidable. Trivially, $(\forall u \in \omega^\omega)$ $(u \in B \Rightarrow u \in W)$.
Let u such that, for each k, $u \frown k \in W$.
If the length of u is at least 1, then $u \frown k$ in W means that an ancestor of $u \frown k$ belongs to B. This cannot be $<>$, since then we would know $\neg[\forall n A(n)]$. Hence, the ancestor of $u \frown k$, which belongs to B, satisfies $A(1_{u \frown k})$. But then, $A(1_u)$ too, and then $u \in B \Rightarrow u \in W$.
If u is root $<>$, then hypothesis $<> \frown k \in W$, for each k, means that $(k) \in W$, which amounts to $(k) \in B$, for each k. This is equivalent to our knowledge of $\forall n A(n)$, which is impossible. Therefore, while hypotheses of CBI are satisfied, its conclusion $<> \in W$ cannot be accepted, because this presupposes our knowledge either of $\forall n A(n)$ or of $\neg[\forall n A(n)]$.$\diamond$

Proposition 13.11: DBI and CCP (Continuity Choice Principle) entail MBI.

Proof: Let B and W properties satisfying MBI. We define predicate $A(\alpha, n)$ by

$$A(\alpha, n) \Leftrightarrow n_\alpha \in B,$$

which is monotone, $(A(\alpha, m)$, if $m > n)$, since B is monotone. Obviously, $(\forall \alpha \in M)(\exists n)A(\alpha, n)$, since $(\forall \alpha \in M)(\exists n)(n_\alpha \in B)$. Applying **CCP** on $A(\alpha, n)$,

$$(\forall \alpha \in M)(\exists n)A(\alpha, n) \Rightarrow (\exists \theta)(\forall \alpha)A(\alpha, \theta(\alpha)),$$

where intuitionistic function θ is activated for the first time by the initial segments of sequences α for which *we know* that belong to B. We also define a new property B' by

$$u \in B' \Leftrightarrow \exists v, v \preceq u : \; v \text{ activates } \theta^*.$$

B' is decidable, since, if u is an -node, starting from u and going back to its ancestors, we check if they activate θ^* or not. Although B may be non-decidable, we formed through B a decidable species of *exactly* those nodes which contain all our knowledge regarding the expression $(\forall \alpha \in M)(\exists n)(n_\alpha \in B)$.
Also, if W is the species in the hypotheses of MBI, then $u \in B' \Rightarrow u \in B$, since there is an ancestor v of u which belongs to B and B is monotone, therefore $u \in W$. Also, $(\forall u \in M)$ $(\forall k(u \frown k \in W) \Rightarrow u \in W)$ is satisfied by the hypothesis of MBI.
Hence, the hypotheses of DBI are satisfied, and the conclusion $<> \in W$ of MBI is derived.$\diamond$

Proposition 13.12: Intuitionistically, $MBI \Rightarrow CBI$ is unacceptable.

Proof: If that implication was intuitionistically true, then by DBI and CCP and the implication $DBI + CCP \Rightarrow MBI$, then $DBI + CCP \Rightarrow CBI$, something which is unacceptable by Proposition 13.10.$\diamond$

Proposition 13.13: The following implications hold:
(i) $DBI \Rightarrow DBT$.

(ii) $MBI \Rightarrow MFT$.
(iii) $CBI \Rightarrow CFT$.

Proof: (i) Let the hypotheses of DFT are given. We define properties B and W as follows:

(I) $u \in B \Leftrightarrow A(u)$.
(II) $u \in W \Leftrightarrow (\exists n)(\forall \alpha \in u)(\exists m \leq n)A(m_\alpha)$.

Then,

$$<> \in W \Leftrightarrow (\exists N)(\forall \alpha \in T)(\exists m \leq N)A(m_\alpha),$$

which is the conclusion of DFT for T.
Trivially, B is decidable, implication $u \in B \Rightarrow u \in W$ is satisfied and also property $(\forall \alpha)(\exists n)n_\alpha \in B$. Let u be a node such that $u \frown k \in W$, for each k. This means that each sequence extending $u \frown k$ satisfies A for some initial segment of length $\leq n_k$, for some bound n_k. Since our spread is a fan, there are finite immediate successor nodes $u \frown k$ of node u. If N is the maximum of all n_k, then $u \in W$, since a sequence extending u extends some node among those finite nodes $u \frown k$, therefore, it satisfies A before its length becomes $N + 1$. As we have already seen, the conclusion of DBI gives directly the conclusion of DFT.
(ii) The proof is similar to that of previous case, defining
(I) $u \in B \Leftrightarrow R(u)$.
(II) $u \in W \Leftrightarrow (\exists n)(n > l(u)) : \ [\forall v \ v \succ u \ \wedge \ l(v) = n] \Rightarrow R(v)$,
where $l(u)$ is the length of node u and R the monotone predicate of hypothesis of MFT. Also,

$$<> \in W \Leftrightarrow (\exists n)(n > 0) : \ [\forall v \ l(v) = n] \Rightarrow R(v).$$

I.e., $(\forall \alpha \in 2^\omega)R(n_\alpha)$, the conclusion of MFT.
Case (iii) is proved exactly like (i).$\diamond$

Based on (i) of previous proposition, DBI with KCP entail KFT (in FIM), using Proposition 13.7.

Let DBI(F), MBI(F) be the induction principle DBI and MBI on fans. What we have actually showed in Proposition 13.13 are the implications

$$DBI(F) \Rightarrow DFT,$$
$$MBI(F) \Rightarrow MFT.$$

The inverse implications also hold, something with special philosophical significance.

Proposition 13.14: The following implications hold:
(i) $DFT \Rightarrow DBI(F)$.
(ii) $MFT \Rightarrow MBI(F)$.

Proof: (i) Let the hypotheses of DBI(F) and A a decidable predicate defined by

$$A(u) \Leftrightarrow u \in B.$$

A obviously satisfies hypothesis $(\forall \alpha \in F)(\exists n)A(n_\alpha)$, hence, by the conclusion of DFT, there is a natural number N, such that all sequences satisfy A before their length becomes $N + 1$. Thus, the nodes of F which satisfy B form a polygon line connecting

the edges of F. If we consider for simplicity (although the argument is the same in the general case too) the fan 2^ω, we reach again the conclusion $<> \in W$.
In case (ii) we find in a similar way a tree leading to $<> \in W$ too.$\diamond$

The last two propositions show that the intuitionistic truth of DBI and MBI are tightly connected with the truth of generalized forms of fan theorem DFT and MFT. It remains to study (see Paragraph 14) the relationship between BFT and its generalizations DFT, MFT, or DBI and MBI.

14. Brouwer's dogma and the theorem of Martino and Giaretta. Brouwer's proof of BFT is generally considered problematic, for not one reason only. There are three main kinds of critique in the literature, and we also offer another one in Paragraph 15. These are:

(I) Critique of the supposedly metamathematical character of Brouwer's proof of BFT because of the analysis of R_u as a tool for the derivation of its conclusion.
(II) Critique of Brouwer's dogma BD. Again BD can be considered of metamathematical character.
(III) Characterization of Brouwer's proof as cyclic, since, through Martino-Giaretta theorem (MGT), Brouwer's assumption BD is equivalent to the inductive argument of his proof.

Regarding (I), Heyting's answer [Heyting 1966 p.45] is the following:

> [But in almost every case it is not the supposed construction itself that plays a part in the proof, but only its result. The new feature in the proof of the fan theorem is, that the possible form of the supposed construction is explicitly involved in it. If we are well aware that the hypothesis of a theorem consists always in the assumption of a previous execution of some construction, we can offer no objection against the use of considerations about the way in which such a construction can be performed as a means of proof.]

This response is all that Heyting offers to objection (I), saying nothing on the question whether it is possible to handle the multiplicity of possible constructions of hypothesis of BFT without incorporating something like BD. As Epple says (in [Epple 1997] pp.164-5):

> This crucial passage (in our language, from the existence of $R_{<>}$ to the existence of a canonical proof of the securability of $<>$) is evidently a meta-mathematical belief, a belief about what it means to have a proof ($R_{<>}$). This leaves Brouwer's whole argument in an awkward position. As long as intuitionistic mathematics is *not* formalized, such a meta-mathematical belief must necessarily remain informal. Hence a crucial theorem of intuitionistic mathematics *depends essentially* on an informal belief. In principle, such a belief does not conflict with the intuitionistic project. But it is legitimate to ask whether the meta-mathematical belief in question is as transparent as Brouwer's epistemological standards require it to be.]

Brouwer considered BD natural, though "as late as 1952 had to admit that a simpler

proof (of BFT) had eluded him"[89]. Also, van Atten[90] doesn't accept Epple's general tenet, believing that Brouwer's proof doesn't betray the intuitionistic principles, without giving though, some arguments in favor of his opinion.
The obscurity of BD justified Kleene's reaction to use the scheme of decidable bar induction (DBI) as an axiom. In case $u \in W$ means that u has the w.o.p. for nodes, then BFT is directly derived. But to prove BFT through an axiom violates basic principles of Brouwer's mature intuitionism framework. Brouwer always searched for a conceptual proof of FT, not an axiomatic one. Kleene justified this seemingly evasion of his to postulate an axiom schema by his independence result of bar induction from the other intuitionistic principles[91]. As we discuss in Paragraph 13, this independence of DBI from the other principles of FIM is an expression of the difference between bar induction and the rest, derived by definitions, "principles" of BIA. This difference though, depends on our analysis of another argument against the intuitionistic validity of Brouwer's proof of BFT, found in Paragraph 15.

The following analysis of BD and MGT is based on the original paper of Martino and Giaretta, slightly adopted to our language. Martino and Giaretta give the following precise inductive definition of a **canonical proof** (c.p).

(i) An η-inference

$$\frac{u\ secured}{u\ securable}$$

is a c.p of "u is B-securable", or for simplicity "u is securable".
(ii) If R_u is a proof of "u is securable", then

$$\frac{R_u}{u \frown k\ securable}$$

is a c.p of "$u \frown k$ is securable", provided $u \frown k \in M$.
(iii) If $R_{u \frown k}$ is a c.p of "$u \frown k$ is securable", then

$$\frac{R_{u\frown 1}, R_{u \frown 2}, \ldots}{u\ securable}$$

is a c.p of "u is securable".

Brouwer's dogma has then the following precise formulation, since c.p is precisely defined:

BD: If <> is B-securable, then there exists a c.p of "<> is B-securable".

If we define the species $IndB$ by:

$$u \in IndB \Leftrightarrow \exists c.p \text{ of "}u \text{ is securable"},$$

then, by (i)-(iii), $IndB$ is the $\eta, \zeta, \digamma$-closure of bar B i.e.,
(I) $u \in B \Rightarrow u \in IndB$.
(II) $u \in IndB \Rightarrow u \frown k \in IndB$.
(III) $\forall k\ u \frown k \in IndB \Rightarrow u \in IndB$.

(I) is justified as follows: Since $u \in B$ an ancestor v of u belongs to B_0 i.e., u is secured, hence, by (i), an η-inference is a c.p of "u is secured". (II) and (III) are direct

[89]In [van Stigt 1990] p.379.
[90]In [van Atten 2004] p.40, p.61.
[91]See [Kleene, Vesley 1965] p.51.

translations of (ii) and (iii) respectively. Hence, Brouwer's dogma has also the following formulation:

BD: If $<>$ is B-securable, then $<> \in IndB$.

As we have seen in Paragraph 13, Kleene's counterexample (KC) refuted Bar theorem without decidability condition. Equivalently, KC refutes bar induction scheme BI, where BI is formulated here (as in [Martino, Giaretta 1979]) as follows:

BI: If $<>$ is B-securable, then B inductively bars $<>$,

where "B inductively bars $<>$" means that $<>$ belongs to $B^{\digamma}$, the η and $\digamma$-closure of B i.e.,

(i) $u \in B \Rightarrow u \in B^{\digamma}$.
(ii) $\forall k \; u \frown k \in M, \; u \frown k \in B^{\digamma} \Rightarrow u \in B^{\digamma}$[92].

As in Paragraph 13, **KC refutes BI:** if ω^{ω} is the spread M and A is a decidable predicate on the naturals i.e., $\forall n(A(n) \vee \neg A(n))$, such that we do not posses yet a proof neither of $\forall n A(n)$, nor of $\neg[\forall n A(n)]$, we then define B by

$$<> \in B \Leftrightarrow \neg[\forall n A(n)] \text{ and } (n) \in B \Leftrightarrow A(n).$$

Obviously, $<>$ is B-securable, since, if α is any sequence, then $A(n) \vee \neg A(n)$, where $(n) = (\alpha(0))$, the 1-segment of α. If $A(n)$, then $(n) \in B$, while, if $\neg A(n)$, then $\neg[\forall n A(n)]$ i.e., $<> \in B$. Hence, each α cuts B.
But $<> \notin B^{\digamma}$, since if that was not the case we would know it through an η or $\digamma$-inference.
If $<>$ belongs to B, then we would know that $\neg[\forall n A(n)]$, which is impossible.
If we have $<> \in B^{\digamma}$ because $\forall n (n) \in B^{\digamma}$, then we reach the impossible knowledge $\forall n A(n)$, since

$$(n) \in B^{\digamma} \Rightarrow (n) \in B.$$

If we suppose $\neg A(n)$, then we get $\neg[\forall n A(n)]$, which is impossible. But, by the decidability of A, we then get $A(n)$ i.e., $(n) \in B$. If that is the case for each n we reach the impossible knowledge $\forall n A(n)$.

The above negation of BI by KC was considered by Dummett, in [Dummett 1977], as a refutation of BD. His point was that since Brouwer's 1924 proof of BFT excludes all ζ-inferences and makes no reference either to the decidability or to the monotonicity of B, it is certainly wrong because of KC. Dummett concluded that what is wrong with it must be BD, the only unjustified assumption of the proof, and he considered KC a refutation of it. Martino and Giaretta criticized Dummett's interpretation of KC, a critique accepted later by Dummett (in [Dummett 2000]).
Martino and Giaretta provided to the hypothesis $<>$ is B-securable a c.p, in accordance to their precise definition of a c.p. This c.p is a description of how $<>$ is B-securable, where B is the one defined in KC, combining $\eta, \digamma$ *and* ζ-inferences.

$<>$ is B-securable because of a $\digamma$-inference, where each (n) is securable, either because (n) cuts B, if $A(n)$, or, if $\neg A(n)$, $<>$ cuts B, therefore, by an η-inference, $<> \in B^{\digamma}$ and then, by a ζ-inference, $(n) \in B^{\digamma}$.

[92] If B is monotone or decidable, we get MBI or DBI respectively.

Therefore, Martino and Giaretta conclude that KC does not refute BD but the unconditional eliminability of the ζ-inferences from a c.p, since the above c.p can't avoid the ζ-inference in case, which cannot be excluded, that $\neg A(n)$, for some n.
This exclusion is possible if hypotheses of monotonicity or decidability are added to B.

Proposition 14.1: (i) If $u \in IndB$ and B is monotone, then $u \in B^{\digamma}$.
(ii) If $<> \in IndB$ and B is decidable, then $<> \in B^{\digamma}$.

Proof: (i) Since (I) and (III) of inductive definition of $IndB$ are same to (i) and (ii) respectively of the inductive definition of $B^{\digamma}$, it suffices to show that $B^{\digamma}$ is ζ-closed i.e., $u \in B^{\digamma} \Rightarrow u \frown k \in B^{\digamma}$. But $u \in B^{\digamma}$ means that $u \in B$, or for each k, $u \frown k \in B^{\digamma}$. If $u \in B$, then, by monotonicity of B, $u \frown k \in B$, hence $u \frown k \in B^{\digamma}$. If for each k, $u \frown k \in B^{\digamma}$, then automatically we get what we want. Thus we have proved that under monotonicity hypothesis of B

$$IndB = B^{\digamma}.$$

(ii) A node u is called *pre-barred by B* iff an ancestor v of u belongs to B. Then, we prove inductively:

$$(*) \quad u \in IndB \Rightarrow u \in B^{\digamma}, \text{ or } u \text{ is pre-barred by } B.$$

By the definition of $IndB$, if $u \in B$, then $u \in B^{\digamma}$ *and* u is pre-barred.
If $u = v \frown k$ and $v \in B^{\digamma}$, or v is pre-barred, then we show the same for u examining each case separately. If $v \in B^{\digamma}$ because $v \in B$, then u is by definition pre-barred. If $v \in B^{\digamma}$ because $v \in B^{\digamma} - B$, then the only way for that is that for each k, $v \frown k \in B^{\digamma}$, therefore, $u \in B^{\digamma}$ too.
If $u \frown k \in IndB$, for each k, then we work as follows: u is pre-barred or not, since we can check in finite time each ancestor v of u, whether it is in B or not, because of the decidability of B. If u is pre-barred, then we have reached our conclusion. If not, then, by the inductive hypothesis, each $u \frown k \in B^{\digamma}$ or it is pre-barred. If $u \frown k$ is pre-barred, for some k, then $u \frown k \in B$, since u is considered not pre-barred. Hence, $u \frown k \in B^{\digamma}$. Therefore, $u \frown k \in B^{\digamma}$, for each k, and then $u \in B^{\digamma}$. Note that in any case we know exactly which part (or parts) of the disjunction is satisfied, as the BHK-interpretation of disjunction demands.
Applying $(*)$ on the root $<>$, we get, by hypothesis, that $<> \in B^{\digamma}$ or $<>$ is pre-barred i.e., $<>$ belongs to B. Therefore, in any case $<> \in B^{\digamma}$.$\diamond$

By BD, if $<>$ is B-securable, then $<> \in IndB$, hence, adding monotonicity (Proposition 14.1(i)) or decidability (Proposition 14.1(ii)) to B, then $<> \in B^{\digamma}$ i.e., $<>$ belongs to the $\eta, \digamma$-closure of B and the ζ-inferences are indeed eliminable.

Proposition 14.2 Martino-Giaretta theorem (MGT): BD is equivalent to MBI.

Proof: Suppose BD and that the root $<>$ is securable under monotone B. By BD, $<> \in IndB$, and by Proposition 14.1(i), $<> \in B^{\digamma}$, since B is monotonic.
Suppose MBI and that $<>$ is B-securable. Let B^{ζ} be the monotonic closure of B i.e.,
(i) $u \in B \Rightarrow u \in B^{\zeta}$.
(ii) $u \in B^{\zeta} \Rightarrow u \frown k \in B^{\zeta}$.

Since $<>$ is B-securable, $<>$ is B^{ζ}-securable too. Since B^{ζ} is, by its definition, a monotone bar, applying MBI on B^{ζ} we get that $<> \in B^{\zeta \digamma}$. However, as we concluded in the proof of Proposition 14.1(i),

$$B^{\zeta^{F}} = IndB^{\zeta^{F}} = IndB,$$

since B^{ζ} is the ζ-closure of B and hence, $B^{\zeta^{F}}$ is the η, ζ and F-closure of B, therefore $B^{\zeta^{F}}$ is identical to $IndB$.$\diamond$

Note that BD entails DBI by direct application of the syllogism of the first implication of the above proof, where monotonicity of B is replaced by its decidability and Proposition 14.1(ii) is used instead of Proposition 14.1(i).
By MGT, Martino and Giaretta concluded that

> [Since BD and MBI are equally reliable, Dummett's claim that it is convenient to assume MBI directly as an axiom, in order to avoid the problematic character of BD, turns out to be quite groundless ... We have two further remarks to make about the idea, suggested by Dummett, that BD would be an ad hoc axiom. First of all BD seems to express correctly Brouwer's intuition ... that the knowledge that $u \notin B$ is barred must be based on an examination of its surrounding nodes. On the contrary this intuition is lost in the bar theorem, in which no explicit reference is made to the predecessor of u. In the second place, even if no more evidence is attributed to BD than to MBI, or one prefers to ignore the problem of evidence, BD seems to us interesting in itself. While, in fact, the bar definition involves the notion of infinite sequence, this notion does not occur at all in the definition of $IndB$, which is stated entirely in terms of finite sequences. Thus, BD says, in effect, that the bar notion, even for a B such that BI does not hold, can be expressed in terms of finite sequences. In this sense, BD can be regarded as a generalization of MBI.]

Although Martino and Giaretta regard BD fairly plausible, provided that B can be defined without reference to the concept of infinite sequence, we see MGT as an indication that if MBI is problematic from the intuitionistic point of view, as we show in the next paragraph, this problematic character of MBI is transferred to BD too.

15. Another argument against the intuitionistic validity of Brouwer's proof of Fan theorem. As we described in previous paragraph, up till now the problem with Brouwer's proof of BFT was mainly centered around the incompatibility of BD with Brouwer's epistemological declarations. Although we agree with Epple on his critique on the transparency of DB, someone could insist, like van Atten, that Brouwer's proof is intuitionistically clear. Brouwer after all, supported the idea that DB is transparent enough or that it is a matter of some kind of intuition in order to be accepted.
In our view, there is a serious flow in Brouwer's argumentation and most important, this flow derives from the intuitionistic point of view. As we have described in Paragraph 11, the main idea of Brouwer's proof of BFT is the following:

Proof R_u of the securability of node u starts with the securability of the elements of B_0 (the securability of the, in principle, infinite nodes $u \notin M$ is not that essential). Through ζ-inferences, and mainly through F-inferences, and since R_u has the preservation property, the w.o.p.n is actually transferred from the premisses nodes to the *conclusion nodes of R_u. Especially, w.o.p.n is transferred to the final conclusion node u of R_u.*

The often discussed quote 7 of Brouwer (in [Brouwer 1927] p.460) is characteristic. According to it, that node u has the w.o.p.n is something that someone can see, if thinks intuitionistically, and the whole proof is not that necessary[93].

> [When carefully considered from the intuitionistic point of view, this securability (he means R_u) is seen to be nothing than but the property (he means the w.o.p.n) defined by the stipulation that it shall hold for every element of B_0[94] and for every inhibited element of $\omega^{<\omega}$[95], and that it shall hold for an arbitrary $(a_1, a_2, ..., a_n)$ as soon as it is satisfied, for every k, for $(a_1, a_2, ..., a_n, k)$. This remark immediately implies the well-ordering property for an arbitrary $(a_1, a_2, ..., a_n)$. The proof carried out in the text for the latter property, however, seems to be of interest nevertheless on account of the propositions contained in its elaboration]

In our view though, Brouwer's description of R_u is not constructively innocent.

The real problem in the proof of BFT is that B_0 may have infinite elements and it is not justified that R_u starts and, most crucially, ends the way Brouwer says it does. If R_u is a man-made proof, as it has to be from the intuitionistic point of view, starting R_u from η-inferences, we may need, in general, absolutely infinite time in order to "go down" to nodes of less and less length and finally to node u. In order to consider R_u completed, Brouwer employs the intuitionistically unaccepted concept of absolute infinity.

We may imagine a thin bar of fan as a line segment, or a polygon line segment connecting the ends of the fan. All such thin bars contain a node of maximum length, hence the above down to u procedure seems plausible. In general case though, a thin bar may "consist" of nodes of ever growing length, without being able to "draw" such a segment. We call such a bar the *(im)possible thin bar* K_0. Brouwers description of the proof $R_{<>}$ bypasses this possibility, which is half of the essence of the problem, without excluding it. Brouwer bypasses this crucial point through an argument, which turns out to be of an absolute infinity character. He cannot start from a finite number of nodes of K_0 and go up to the root, since it is possible to need the securability of a K_0-node of higher level and so on. E.g., securability of a node u_1 may need the fact that $v_1 \in K_0$, while securability of a node u_2, may need the fact that $v_2 \in K_0$ and the securability of node u_3 etc.
Hence, Brouwer needs, in principle, all K_0-nodes to go up to the root, which is a use of absolute infinity. So, the supposable proof $R_{<>}$ is not, in general, a man-made proof, since it is not applied to, at the beginning possible, bar K_0. Brouwer acts as if he knows that K_0 is impossible, without proving though its impossibility.
Hence, in our view, proof $R_{<>}$ starts and finishes the way Brouwer says, if it is possible to consider all η-inferences together. That means:

(i) either there is a uniform way of description of all η-inferences i.e., of B_0,
(ii) or there is a tacit use of absolute infinity, considering that going further and further

[93] Kleene also agrees with this footnote.
[94] We have replaced Brouwer's original notation with the one we used in Paragraph 11.
[95] I.e., for each node which is not M-accepted.

from the root finally stops and reaching the root follows, in any case[96].
(i) would mean though, that there is an extra condition in BBT_1, something which is never mentioned.
(ii) would mean a serious deviation from the intuitionistic point of view.
Even if M is a fan and $B = B_\varphi$ is an infinite species, then, at the beginning of $R_{<>}$, we do not know that B_0 is a finite species. $R_{<>}$ starts with inferences

$$\frac{u\ is\ secured}{u\ is\ securable}$$

but we need to know that these inferences are finite, or some other uniform generation of B_0 is needed, in order to start $R_{<>}$. Hence, *the real problem with Brouwer's proof of BFT is that he considers a line of progression in $R_{<>}$, without justifying constructively that such a proof is completed in finite time.*
Brouwer proves the finiteness of B_0 only on the hypothesis that $R_{<>}$ is completed and proceeds the way he describes. But before starting the proof, Brouwer doesn't know that B_0 is finite and he does not explain how we may go from the η-inferences down to the root $<>$. So, the proof $R_{<>}$ that Brouwer adjusts to hypothesis P' is not an intuitionistic mathematical object.

Even if ζ-inferences are eliminated (see Proposition 14.1), still there is no guarantee that we can reach the securability of $<>$ using only η and $\digamma$-inferences, starting from the elements of B_0, since the nodes of B_0 needed for the securability of a certain node may have unbounded length.
Although we elaborated the above argument independently from Epple, it can be found in a condensed form in [Epple 1997] p.166, without Epple reaching though, the same conclusions on the unacceptability of bar induction. Epple writes:

> [The striking feature of bar induction is that, in contrast to ordinary complete induction along the sequence of natural numbers, its *anchoring* is not completely transparent. How can we be sure that the hierarchy of induction steps ever reaches the root of a given fan? In fact, how can we be sure that even a *single* induction step is warranted? Does this not presuppose that we already *know* that all necessary anchoring statements of type (η-inferences) are within finite reach?]

A serious consequence of our analysis of Brouwer's proof of BFT is that the non-constructive character of Brouwer's argument applies to DBI too. While in standard induction there is a fixed beginning, in DBI there is a fixed end without any explication of how the η and $\digamma$-inferences it employs lead to the w.o.p. of $<>$. This is related to the result of Martino and Giaretta, that BD is equivalent to DBI, assuming CP, or directly equivalent to MBI.
Our analysis of Brouwer's inductive argument, or DBI, reveals a tacit use of absolute infinity or a tacit use of an extra condition of uniform knowledge of B_0. Through MGT this pathology is transferred to BD too. Therefore, a whole new foundation of intuitionistic analysis is needed, independently from DBI or MBI. This has extreme

[96] In [Brouwer 1981] p.101, van Dalen refers to the induction Brouwer employs to his proof as "transfinite induction". Although, we are not sure what exactly van Dalen means, it seems a characterization of Brouwer's induction close to ours.

consequences. BFT has to be proved in a different way. Of course, first we have to explain why BFT should be proved i.e., why it must reflect an intuitionistic truth. Even if that is accomplished, DBI, or MBI, as an intuitionistic tool, doesn't seem right to use. This has serious consequences in the proof of intuitionistic theorems considered to be proved through DBI (e.g., intuitionistic Ramsey theorem is proved in [Veldman, Bezem 1993] through DBI). As a results of our analysis though, intuitionistic truths, DBI or MBI excluded, need to be completed in a new way.
Epple, in [Epple 1997] concludes:

> [Even today, it appears to be an open question whether a non-circular justification of the form of transfinite induction proposed by Brouwer can be found that satisfies the standards of strict verificationism. ... In the end ... it was not so much Brouwer's *mathematics* that was problematic, but rather, despite all its potential philosophical merits, the *epistemology* that formed a crucial part of his project and that guaranteed, according to Brouwer, the undeniable superiority of his theory over the rest of modern mathematics.]

In our view, preserving, or even strengthening Brouwer's epistemology may provide a way out to the dead end formed by his mathematics (bar induction) and its incompatibility to his epistemological standards.

Appendix

A classical presentation of Baire space $\mathcal{N}$ and Cantor space $\mathcal{C}$. From the point of view of descriptive set theory the space of real numbers has three faces: the set of real numbers $\mathbb{R}$, Baire space $\mathcal{N}$ and Cantor space $\mathcal{C}$. Levy accurately names these spaces *the real spaces* (in [Levy 1979]).
$\mathbb{R}$, $\mathcal{N}$ and $\mathcal{C}$ are not homeomorphic to each other and behave, generally differently. E.g., while $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}$, the cartesian products of Baire and Cantor space are homeomorphic to themselves i.e., $\mathcal{N}$ and $\mathcal{C}$ are "dimensionless". Also, $\mathbb{R}$ is connected, while Baire and Cantor spaces are totally disconnected.
On the other hand, their differences are in some sense negligible, while the structure of $\mathcal{N}$ and $\mathcal{C}$ is simpler than that of $\mathbb{R}$. E.g., Baire space is homeomorphic to the irrationals, and also it is homeomorphic to the finally non-constant sequences of $\mathcal{C}$. Cantor space is homeomorphic to Cantor set and it is standardly surjected to $[0, 1]$.
Many of their common properties are shared by the completely metrizable metric spaces (Polish spaces). At the end, one can study the space of reals that fits to his ends.
The study of Baire and Cantor spaces gives us the opportunity to compare concepts and results between classical and intuitionistic analysis. Baire space appears in Brouwer's analysis, not in the usual set-theoretical context, but constructively interpreted through the concept of universal spread. Hence, the same space, but differently interpreted, is the ambient space of the intuitionistic continuum. Although the mathematical scene looks the same, the foundational differences between classical and intuitionistic analysis are responsible for the differences between the classical and the intuitionistic Baire space.
In this Appendix we present the basic properties of classical Baire and Cantor space working in the usual set-theoretical context[97].

Baire space $\mathcal{N}$ is the set of sequences of natural numbers and it can be seen as the body of the infinitely branching pruned Baire tree on $\mathbb{N}$ (see Paragraph 2). On the cartesian product of $\mathcal{N}$ with itself the following metric is defined: if $\alpha, \beta \in \mathcal{N}$, then

$$\rho(\alpha, \beta) = \begin{cases} \frac{1}{\eta(\alpha,\beta)} & , \text{if } \alpha \neq \beta \\ 0 & , \text{if } \alpha = \beta \end{cases}$$

where $\eta(\alpha, \beta)$ is the least natural number k (0 is not included in $\mathbb{N}$), for which $\alpha_k \neq \beta_k$, if $\alpha \neq \beta$. I.e., $\eta(\alpha, \beta)$ is the minimum index for which α, β differ[98].

Proposition A.1: ρ is a metric on $\mathcal{N}$.

Proof: By the definition of ρ

$$0 \leq \rho(\alpha, \beta) \leq 1.$$

and $\rho(\alpha, \beta) = \rho(\beta, \alpha)$. We see that ρ expresses the following intuitively expected fact: the bigger $\eta(\alpha, \beta)$ gets, i.e., the bigger the common initial segment of sequences α and β gets, the smaller is their distance. To the limit, when $\alpha = \beta$, then formally $\eta(\alpha, \beta) = \infty$ and $\rho(\alpha, \beta) = \frac{1}{\infty} = 0$.

[97] For all the concepts and results that we refer to without proof see any standard book on topology, e.g., [Dugundji 1989].

[98] Intuitionistically speaking, we cannot even state the above definition, since the least number principle does not hold on subspecies of the species of natural numbers.

For the proof of the triangle inequality

$$\rho(\alpha,\beta) \leq \rho(\alpha,\gamma) + \rho(\gamma,\beta)$$

we consider the following cases:
(A) If $\alpha = \beta$, then $\rho(\alpha,\beta) = 0$ and $\rho(\alpha,\gamma) + \rho(\gamma,\beta) \geq 0$, therefore the inequality trivially holds.
B_1) If $\alpha \neq \beta$ and $\eta(\alpha,\gamma) < \eta(\alpha,\beta)$, then $\frac{1}{\eta(\alpha,\beta)} > \frac{1}{\eta(\alpha,\gamma)}$ and so $\frac{1}{\eta(\alpha,\beta)} < \frac{1}{\eta(\alpha,\gamma)} + \frac{1}{\eta(\beta,\gamma)} = \frac{2}{\eta(\alpha,\gamma)}$, since $\eta(\alpha,\gamma) = \eta(\beta,\gamma)$ and γ differs from α and β on the same index.
B_2) If $\alpha \neq \beta$ and $\eta(\alpha,\gamma) > \eta(\alpha,\beta)$

$$\begin{array}{c}
\gamma_1 = \alpha_1 = \beta_1 \\
. \quad . \quad . \\
. \quad . \quad . \\
. \quad . \quad . \\
\gamma_{k-1} = \alpha_{k-1} = \beta_{k-1} \\
\gamma_k = \alpha_k \neq \beta_k \\
. \quad . \quad . \\
. \quad . \quad . \\
. \quad . \quad . \\
\gamma_{l-1} = \alpha_{l-1} \\
\gamma_l \neq \alpha_l
\end{array}$$

But then, $\eta(\gamma,\beta) = \eta(\alpha,\beta)$, hence $\frac{1}{\eta(\alpha,\beta)} = \frac{1}{\eta(\gamma,\beta)}$, so again we take, $\frac{1}{\eta(\alpha,\beta)} < \frac{1}{\eta(\alpha,\gamma)} + \frac{1}{\eta(\gamma,\beta)}$.
B_3) If $\alpha \neq \beta$ and $\eta(\alpha,\gamma) = \eta(\alpha,\beta)$, then $\frac{1}{\eta(\alpha,\beta)} = \frac{1}{\eta(\alpha,\gamma)}$ and so,

$$\frac{1}{\eta(\alpha,\beta)} < \frac{1}{\eta(\alpha,\gamma)} + \frac{1}{\eta(\gamma,\beta)}$$

Actually, we saw that if $\gamma \neq \alpha$ and $\gamma \neq \beta$, then the triangle inequality holds strictly. Equality holds iff $\gamma = \alpha$ or $\gamma = \beta$.$\diamond$

Since $\mathcal{N}$ is not a vector space, ρ is not a norm generated metric.
It is easy to see checking each one of the cases of the above proof that

$$\eta(\alpha,\beta) \geq min(\eta(\alpha,\gamma), \eta(\gamma,\beta))$$

therefore, ρ satisfies the stronger to triangular inequality, hypermetric triangular inequality

$$\rho(\alpha,\beta) \leq max(\rho(\alpha,\gamma), \rho(\gamma,\beta))$$

which holds in p-adic metrics, where p is prime, and they are defined on the rationals $\mathcal{Q}$. The larger is the power of p which divides the difference of two rationals, the "closer" they get with respect to a p-metric. These p-metrics are generated by the corresponding p-norms, for which

$$|x+y| \leq max(|x|, |y|)$$

holds. A norm satisfying the above inequality is called non-Archimedean and the corresponding metric, non-Archimedean metric [99]. Furthermore, each non-trivial norm on rationals is equivalent to a p-norm or to the usual Archimedean norm on them (Ostrowski's theorem).
But p-norms and p-metrics behave quite differently from the usual norms and metrics. E.g., it is easy to see that a triangle in a field F with a non-Archimedean norm is always an isosceles one and any point of the open disc $B(\alpha, \varepsilon) = \{x \in F, |x - \alpha| < \varepsilon\}$ can be considered a center of it (see [Koblitz 1977] p.6).
In Baire space:

$$B(\alpha, \frac{1}{k}) = \{\beta \in \mathcal{N}, \rho(\alpha, \beta) < \frac{1}{k}\} = \{\beta \in \mathcal{N}, \eta(\alpha, \beta) > k\} \cup \{\alpha\}$$

i.e., the open disc with center α and of radius $1/k$, is the set of all sequences β which share the same k-initial segment with α (k-equal). Obviously,

$$B(\alpha, \frac{1}{k}) = B(\beta, \frac{1}{k})$$

for each $\beta \in (\alpha, \frac{1}{k})$. Hence, in the open disc with center α and of radius $1/k$ each element of the disc can be considered a center of it. The non-Archimedean metric of Baire space is the origin of this property of $\mathcal{N}$. Although ρ is the standard metric on $\mathcal{N}$, Baire space can be embedded homeomorphically to $\mathbb{R}$ having a topological behavior based on the Archimedean metric of $\mathbb{R}$.

Proposition A.2: If $(\alpha_n)_n$ is a sequence of elements of $\mathcal{N}$, the following are equivalent:
(i) $\alpha_n \xrightarrow{\rho} \alpha$
(ii) $\forall k\ \exists n, n = n(k)\ \forall n,\ n \geq n(k),\ \alpha_n^k = \alpha^k$

Proof: (ii) $\Rightarrow$ (i) (ii) expresses the fact that after some index n, which depends on k, all sequences have the same k-term with α. That is,

$$\begin{array}{llll} \text{for } k=1 & \exists n(1) & n \geq n(1) & \alpha_n^1 = \alpha^1 \\ \text{for } k=2 & \exists n(2) & n \geq n(2) & \alpha_n^2 = \alpha^2 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \text{for } k & \exists n(k) & n \geq n(k) & \alpha_n^k = \alpha^k \end{array}$$

So, for $n > max\{n(1), n(2), ..., n(k)\}$ all terms α_n have the same initial k-segment with α, therefore, since k is any natural, $\alpha_n \xrightarrow{\rho} \alpha$.
(i) $\Rightarrow$ (ii) Hypothesis $\alpha_n \xrightarrow{\rho} \alpha$ by definition means that

$$(1) \quad \forall k\ \exists n(k) :\ \forall n,\ n \geq n(k),\ \rho(\alpha_n, \alpha) < \frac{1}{k}.$$

[99] We can mimic, somehow, the construction of the algebraically closed and Cauchy-complete (with respect to the usual metric) field of complex numbers $\mathbb{C}$ from the Archimedean field $\mathbb{R}$ and to construct set-theoretically the least field containing the rationals which is algebraically closed and Cauchy-complete with respect to p-norm. This gigantic field is not, in contrast to $\mathbb{C}$, a locally compact space and it is the ambient space of p-adic analysis (for this construction see, for example, [Koblitz 1977] chapters 1 and 3.)

But, if $\alpha_n \neq \alpha$, $\rho(\alpha_n, \alpha) < \frac{1}{k}$ means that $\eta(\alpha_n, \alpha) > k$ and $\alpha_n = \alpha$, by the definition of η.$\diamond$

Proposition A.3: $\mathcal{N}$ is a complete metric space.

Proof: Let $(\alpha_n)_n$ be Cauchy sequence of elements of $\mathcal{N}$, i.e.,

$$(2) \qquad \forall k\ \exists n(k):\ \forall n, m,\ n, m \geq n(k),\ \rho(\alpha_n, \alpha_m) < \frac{1}{k}$$

and if $\alpha_n \neq \alpha_m$,

$$(3) \qquad \rho(\alpha_n, \alpha_m) < \frac{1}{k} \Leftrightarrow \eta(\alpha_n, \alpha_m) > k \Rightarrow \alpha_n^k = \alpha_m^k$$

that is, all sequences α_n, with $n \geq n(k)$ share $\alpha_n^1, \alpha_n^2, ..., \alpha_n^k$ as a k-initial segment. In that way, for each k, the initial k-segment $\alpha_n^1, \alpha_n^2, ..., \alpha_n^k$ of the sequence-limit of $(\alpha_n)_n$ is defined. So, if we define

$$\alpha(k) = \alpha_n^k,$$

then, by Proposition A.2, we get that $\alpha_n \xrightarrow{\rho} \alpha$.$\diamond$

As a complete metric space $\mathcal{N}$ satisfies Baire category theorem i.e., the intersection of a sequence of dense and open subsets of $\mathcal{N}$ is dense in $\mathcal{N}$. Also, $\mathcal{N}$ is a metric space of second category, i.e., $\mathcal{N}$ is not equal to the union of a sequence of nowhere dense subsets of it.
$\mathcal{N}$'s metric is directly connected to the product metric.
If we define on $\mathbb{N}$ the discrete metric

$$\delta(n, m) = \begin{cases} 1 & , \text{if } n \neq m \\ 0 & , \text{if } n = m \end{cases}$$

and since $\delta(n, m) \leq 1$, the product metric is defined on $\mathcal{N}$

$$\sigma(\alpha, \beta) = \sum_{n=1}^{\infty} \frac{\delta(\alpha_n, \beta_n)}{2^n},$$

since $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$. It is easy to see that convergence in product metric is equivalent to pointwise convergence i.e.,

$$(4) \qquad \alpha_n \xrightarrow{\sigma} \alpha \Leftrightarrow (\forall k)\ \ \alpha_n^k \xrightarrow{\delta} \alpha^k$$

Metrics ρ and σ on $\mathcal{N}$ are equivalent i.e., they generate the same family of open sets, or equivalently

$$(5) \qquad \alpha_n \xrightarrow{\sigma} \alpha \Leftrightarrow \alpha_n \xrightarrow{\rho} \alpha.$$

(5) is proved by (4), since if $0 < \varepsilon < 1$,

$$\alpha_n^k \xrightarrow{\delta} \alpha^k \Leftrightarrow \exists n_0\ \ n \geq n_0\ \ \delta(\alpha_n^k, \alpha^k) < \varepsilon \Leftrightarrow \delta(\alpha_n^k, \alpha^k) = 0 \Leftrightarrow \alpha_n^k = \alpha^k$$

and Proposition A.2.$\diamond$

The use of metric σ on $\mathcal{N}$ has a conceptual advantage over the use of metric ρ. A classical sequence is a mathematical object that exists as a whole independently from our knowledge of it. In the definition of ρ the distance between two sequences α, β is determined by an initial segment, the length of the maximum common initial segment of α, β. The rest of the sequences α, β plays no role in the measure of their between distance. As we have seen, this attitude is better suited to intuitionistic sequences of which, generally, we know only an initial segment. The classical conception of sequence is better reflected in the definition of metric σ. Another expression of σ is

$$\sigma(\alpha, \beta) = \begin{cases} \sum_{\alpha(n) \neq \beta(n)} 2^{-n} & , \text{if } \alpha \neq \beta \\ 0 & , \text{if } \alpha = \beta. \end{cases}$$

σ takes into account the whole of sequences α, β, which is known, within mathematical realism. Also, σ can be seen as the limit of a metric on the finite nodes of the tree $\mathcal{N}$, in a more natural way than ρ. I.e.,

$$\sigma(\alpha, \beta) = lim_n \sigma^*(n_\alpha, n_\beta)$$

where

$$\sigma^*(n_\alpha, n_\beta) = \sum_{\alpha(k) \neq \beta(k)}^{n} 2^{-k}$$

and n_α, n_β are the n-initial segments of α, β respectively. On the other hand, metric ρ is easier to handle in the proofs of the topological properties of $\mathcal{N}$.

A (classical) *spread* is a closed subset of $\mathcal{N}$.

Proposition A.4 (Characterization of closed subsets of $\mathcal{N}$: The following are equivalent:
(i) M is a classical spread.
(ii) If α_n a sequence of elements of M such that $\alpha_n \xrightarrow{\rho} \alpha$, then $\alpha \in M$
(iii) M satisfies

$$(6) \qquad (\forall \alpha)(\alpha \in \mathcal{N})(\forall k)(\exists \beta)(\beta \in M)(i < k \rightarrow \alpha(i) = \beta(i)) \Rightarrow \alpha \in M.$$

Property (iii) expresses the fact that M contains all sequences α for which the $(k-1)$-initial segment is the $(k-1)$-initial segment of an element of M.
(iv) M is the body of a tree on $\mathbb{N}$.

Proof: We do not show here the standard proof of the equivalence between (i) and (ii).
(iii) $\Rightarrow$ (ii) If $\alpha_n \xrightarrow{\rho} \alpha$, then, by Proposition A.2,

$$(7) \qquad \forall k \; \exists n(k) : \; \forall n, \; n \geq n(k), \; \alpha_n^k = \alpha^k$$

If $n = max\{n(1), n(2), ..., n(k-1)\}$, α_n is an element of M and it has the same $(k-1)$-initial segment with α. Then, by (iii), $\alpha \in M$.
(ii) $\Rightarrow$ (iii) For each k there exists, by the hypothesis of (6), an element of M, denoted by $\beta(k)$, which is $(k-1)$-same to α (we use here the Principle of Countable Choice). The sequence $(\beta_k)_k$ of elements of trivially satisfies (7), therefore, $(\beta_k)_k \xrightarrow{\rho} \alpha$, and so, $\alpha \in M$.
(iii) $\Rightarrow$ (iv) By Proposition 1.2.1(ii), we know that $M \subseteq [M^*]$, where $M^* = \{n_\alpha | n \in \mathbb{N} \; \alpha \in M\}$. To show the inverse, if α is in $[M^*]$, then $n_\alpha \in M^*$ for each n, so, by (iii), $\alpha \in M$.
(iv) $\Rightarrow$ (iii) By Proposition 1.2.1(iii), $M = [M^*]$. If α is a sequence satisfying the hypothesis of (iii), then $n_\alpha \in M^*$, for each n. Therefore, α is in $[M^*]$ i.e., α is in M.$\diamond$

Since completeness is inherited to the closed subsets of a complete metric space, a (classical) spread is also complete.
With respect to an open ball of $\mathcal{N}$, $B(\alpha, \frac{1}{k})$, which is defined by

$$B(\alpha, \frac{1}{k}) = \{\beta \in \mathcal{N}, \rho(\alpha,\beta) < \frac{1}{k}\} = \{\beta \in \mathcal{N}, \eta(\alpha,\beta) > k\} \cup \{\alpha\}$$

containing all the k-same to α sequences, and we denote for simplicity by $B_k(\alpha)$, we have the following proposition.

Proposition A.5: (i) $B_k(\alpha)$ is a closed set, therefore it is a clopen set.
(ii) $B_k(\alpha)$ is not a compact subset of $\mathcal{N}$.
(iii) $\mathcal{N}$ has no non-empty, clopen, compact subset.
(iv) $\mathcal{N}$ is not locally compact, therefore it is not homeomorphic to $\mathbb{R}$.
(v) The family $B_k(\alpha)$ is a base for the topology of $\mathcal{N}$, and hence, $\mathcal{N}$ is zero-dimensional space. As such, a space $\mathcal{N}$ is totally disconnected.
(vi) $\mathcal{N}$ is separable and consequently second countable space. Moreover, it has a denumerable base of clopen sets.

Proof: (i) Let γ a sequence each n-initial segment of which is the n-initial segment of a sequence β in $B_k(\alpha)$. So, its k-segment is the k-segment of an element of $B_k(\alpha)$, therefore, γ is k-same to α. So, $\gamma \in B_k(\alpha)$ and $B_k(\alpha)$ is a closed set by Proposition A.4.
(ii) Let $\beta \in B_k(\alpha)$ and $B_{k+1}(\beta)$ all $(k+1)$-same to β sequences of $\mathcal{N}$, which are obviously k-same to α, therefore, they belong to $B_k(\alpha)$. These sets ia cover of $B_k(\alpha)$ i.e.,

$$\bigcup_{\beta \in B_k(\alpha)} B_{k+1}(\beta) = B_k(\alpha)$$

If $B_k(\alpha)$ was compact, then this cover would have a finite subcover. I.e., each k-same to α sequence would be $(k+1)$-same to a finite number of sequences extending the k-initial segment of α, which is absurd, since the k-initial segment of α has an infinite number of immediate successors in $\mathcal{N}$.
(iii) Let K be a non-empty, clopen and compact subset of $\mathcal{N}$. Since K is open, for each $\alpha \in K$ there exists $B_k(\alpha) \subset K$. Then $B_k(\alpha)$ itself is compact, as a closed subset of a compact metric space, which is absurd by (ii).
(iv) If $\mathcal{N}$ was locally compact, then each element of $\mathcal{N}$ would belong to some compact, open neighborhood of it, which is absurd by (iii). So, while spaces $k^{\mathbb{N}}$, with $k \in \mathbb{N}$, i.e., the set of sequences with values in $k = \{0, 1, 2, ..., k-1\}$, are all compact, $\mathcal{N}$ is not even locally compact.
If $\mathcal{N}$ and $\mathbb{R}$ were homeomorphic, then their between homeomorphism $\theta : \mathbb{R} \to \mathcal{N}$ would be a continuous, open and onto the Hausdorff space (as a metric space) $\mathcal{N}$ function. Then, $\mathcal{N}$ would also be locally compact[100], which is absurd.
(v) The fact that the family of $B_k(\alpha)$ is a base for the topology of $\mathcal{N}$ is self-evident. Therefore, $\mathcal{N}$ is a 0-dimensional space, as a metric space (hence T_1-space) with a base of clopen sets. As a 0-dimensional space, $\mathcal{N}$ is also totally disconnected i.e., each

[100]Generally, the continuous image of a locally compact space is not locally compact, though if $\theta : X \to Y$ is a continuous, open and onto Y function, where X is locally compact and Y is Hausdorff, then Y is locally compact (se e.g., [Negrepontis et.al. 1988] p.399).

connected component is a singleton[101].

0-dimensional $(X) \Rightarrow$ totally disconnected (X): If $c \neq x$ and $c \in C$, where C is a connected component of X such that $C \supset \{x\}$, then $X - \{x\}$ is open in X. So, there is a clopen subset K of X such that $c \in K$ and $K \supset X - \{x\}$. Clearly $K \cap C$ is a clopen subset of C, which is neither $\emptyset$, since $c \in K \cap C$, nor C, since $x \notin K \cap C$. So, there is a clopen set in the connected space C other than $\emptyset$ or C, which is absurd.

The definitional property of a 0-dimensional space is the starting point of the definition of topological dimension[102].

(vi) Let D_k be the set of sequences with zero tail after their k-initial segment. I.e.,

$$D_k = \{\alpha \mid \alpha \in \mathcal{N}, \forall i \; i > k, \alpha(i) = 0\}$$

The set

$$D = \bigcup_{k \in \mathbb{N}} D_k$$

of all sequences of $\mathcal{N}$ with zero tail (i.e., of all finally constant zero sequences) is denumerable and dense in $\mathcal{N}$. Denumerability is established by the equipollence of D with the set of finite sequences of $\mathbb{N}$, which is denumerable as a countable union of denumerable sets. Density is due to the fact that there is an element of D in any basic set $B_k(\alpha)$ (just add to the k-initial segment of α the constant sequence $\overline{0}$).

Hence, $\mathcal{N}$ is separable and as a separable metric space $\mathcal{N}$ is second countable. The family of $B_k(d)$, with $d \in D$ and $k \in \mathbb{N}$, is a countable base of clopen sets for the topology of $\mathcal{N}$.$\diamond$

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[101] The inverse is not true (see e.g., Arens rectangle in [Steen, Seebach 1978]) but in a compact space the two concepts are equivalent (a proof can be found e.g., in [Walker 1974] p.46). Walker also shows the connection between these very non-connected spaces with the Boolean algebra of their clopen subsets. In a connected space, like $\mathbb{R}$, the only clopen subsets are the empty set and space itself on which the trivial Boolean algebra $\{0, 1\}$ corresponds to. In totally disconnected or 0-dimensional spaces there exist many clopen subsets. If X is totally disconnected and not connected space, then it can be written as $X = A \cup B$, where A, B are clopen in X. And if one of A, B has at least two elements, then it can also be written as the union of two clopen subsets in X and so on. The following Stone's results are fundamental:
(a) A distributive lattice can be represented as the family of clopen subsets of a totally disconnected space.
(b) The Boolean algebra of clopen subsets of a 0-dimensional space X is complete iff X is extremely disconnected (i.e., X is 0-dimensional space and the closure of an open set of X is also an open set).
(c) (Representation theorem of Boolean algebras) A Boolean algebra is isomorphic to the Boolean algebra of clopen subsets of a compact totally disconnected space.

[102] The following inductive definition justifies the term 0-dimensional space[103]:

(a) $dim\emptyset = -1$
(b) The topological dimension a non-empty topological space is given by

$$dimX = sup\{dim_p X, \; p \in X\}$$

(c) If X is a non-empty topological space, then the dimension $dim_p X$ of X on $p \in X$ is given by:

$$dim_p X = 1 + sup \; \{ \; n \mid p \text{ has base with boundary dimension } \leq n \; \}$$

A totally disconnected space X has topological dimension 0. A point $p \in X$ has neighborhood base with clopen sets. Each clopen set has boundary the empty set, which is of dimension -1. So, by (b), $dimX = 0$.

Proposition A.6 (Characterization of compact subsets of $\mathcal{N}$): The following are equivalent:
(i) K is a compact subset of $\mathcal{N}$.
(ii) K is the body of a (pruned) fan F on $\mathbb{N}$.
(iii) K is a classical spread and for all α in K, there is β in $\mathcal{N}$ such that $\alpha(n) < \beta(n)$, for all n.

Proof:(i) $\Rightarrow$ (ii) Since K is closed, it is the body of a tree T. Let $u \in T$. We show that u has a finite number of immediate successor nodes.
The family of all $B(u \frown k)$, where $B(u)$ is the basic clopen set of all sequences α such that $u \prec \alpha$, and $u \frown k \in T$, together with all sets $B(w)$, where $l(w) \leq l(u)$ and $w \in T$, is an open cover of K. Since K is compact, this cover has a finite subcover, therefore, u has a finite number of immediate successor nodes.
(ii) $\Rightarrow$ (i) A topological space X is called *Lindelöf* iff each open cover of X has a countable subcover. It is easy to see that each second countable space is Lindelöf and that a closed subspace of a Lindelöf space is also Lindelöf. Since $\mathcal{N}$ is second countable (Proposition A.5(vi)), $\mathcal{N}$ is Lindelöf, and since K is a compact subset of $\mathcal{N}$, K is Lindelöf too.
Therefore, it is only necessary to show that each countable cover of K has a finite subcover.
Let $K = \bigcup_{i \in I} U_i^*$, where $|I| = \aleph_0$, a countable cover of K. Without loss of generality take $I = \mathbb{N}$. U_i^* is open in K iff $U_i^* = U_i \cap K$ for some U_i open in $\mathcal{N}$. Likewise, a basic clopen set $B^*(u)$ in K, where $B^*(u) = \{\alpha | \alpha \in K : u \prec \alpha\}$, is identical to $B(u) \cap K$. Let E be an open set in K.

$\alpha \in E \Leftrightarrow (\exists u)(u \prec \alpha) : B^*(u) \subseteq E$
$\alpha \notin E \Leftrightarrow (\forall u)(u \prec \alpha)\ B^*(u) \nsubseteq E$
$\qquad \Leftrightarrow (\forall u)(u \prec \alpha)\ (\exists \beta)(u \prec \beta) : \beta \notin E$

We define the following sequence of open sets of K:
$E_1 = U_1^*$.
$E_2 = U_1^* \cup U_2^*$.
....
$E_i = \bigcup_{j=1}^{i} U_j^*$.
Obviously, $i < k \Rightarrow E_i \subseteq E_k$ and $K = \bigcup_{i \in I} E_i$.
We prove the existence of a finite subcover by reductio ad absurdum.
If there is no finite subcover of $K = \bigcup_{i \in I} U_i^*$, or equivalently of $K = \bigcup_{i \in I} E_i$, and since $K = [F]$, we define the following subsets of F:
$V_1 = \{u | u \in F : l(u) = 1\ \wedge\ B^*(u) \nsubseteq E_1\}$,
$V_2 = \{u | u \in F : l(u) = 2\ \wedge\ B^*(u) \nsubseteq E_2\}$,
....
$V_i = \{u | u \in F : l(u) = i\ \wedge\ B^*(u) \nsubseteq E_i\}$, and

$$V = \bigcup_{i \in I} V_i.$$

Each V_i is non-empty, otherwise each E_i would be a finite subcover. In order to show that V is a subtree of F, we need to prove that if $w \in V$ and $u \prec w$, then $u \in V$.

Suppose that $u \notin V$.

$$u \notin V \Rightarrow u \notin V_i, \forall i \Rightarrow u \notin V_{l(u)} \Leftrightarrow B^*(u) \subseteq E_{l(u)}$$

But then, $B^*(w) \subseteq E_{l(u)}$, therefore, $B^*(w) \subseteq E_{l(w)}$, since $E_{l(u)} \subseteq E_{l(w)}$. But this contradicts the fact that $w \in V$ iff $B^*(w) \nsubseteq E_{l(w)}$.
Since V is trivially infinite, applying König's lemma on V, there is an infinite branch α in K which also belongs to $[V]$. We show that $\alpha \notin \bigcup_{i \in I} E_i$.
Suppose the inverse, then
$\alpha \in \bigcup_{i \in I} E_i \Rightarrow \alpha \in E_i$, for some $i \Rightarrow (\exists u \prec \alpha) B^*(u) \subseteq E_i$.
Hence, for each w such that $l(w) > i$ and $u \prec w \prec \alpha$,

$$B^*(w) \subseteq B^*(u) \subseteq E_i \subseteq E_{l(w)},$$

which is absurd, since $w \in V \Rightarrow w \in V_{l(w)}$, where $B^*(w) \nsubseteq E_{l(w)}$.
So, using König's lemma on V we found that the initial cover of K was not actually a cover. Therefore, our basic assumption, that there is no finite subcover of the initial cover which generated the construction of V, is false, and we conclude that K is compact.
(ii) $\Rightarrow$ (iii) Since $K = [F]$ and F is a fan, there is for each n a finite number of F-nodes of length n. Let $u_1, u_2, ..., u_{k(n)}$ be these nodes and $m_1, m_2, ..., m_{k(n)}$ their n-terms. We define $\beta(n)$ to be a natural number greater than all $m_1, m_2, ..., m_{k(n)}$.
(iii) $\Rightarrow$ (ii) Since $K = [T]$, we need to show that T is a fan. If not, then a node u of T would have infinite immediate successor nodes, the $(l(u) + 1)$-terms of which form an unbounded subset of $\mathbb{N}$, a fact which contradicts the hypothesis of (iii).$\diamond$

Corollary: $\mathcal{N}$ is not a K_σ space i.e., $\mathcal{N}$ is not the countable union of compact sets[104].

Proof: The finite countable case is a simple generalization of the argument in the above (ii) $\Rightarrow$ (iii) proof. The infinite countable case though, requires a simple diagonal argument, as it is customary in the transition from the finite case to the infinite countable case.
If $\mathcal{N} = \bigcup_{i=1}^{\infty} K_i$, we construct an element α of $\mathcal{N}$ which cannot be found in any K_i. Since K_1 is compact subset of $\mathcal{N}$, there is an element α^1 of $\mathcal{N}$, not in K_1, such that $\beta(n) < \alpha^1(n)$ for each n. In the same way to K_i corresponds a sequence α^i. Hence, the sequence

$$(\alpha^1(1), \alpha^2(2), ..., \alpha^i(i), ...)$$

belongs to $\mathcal{N}$ but not in $\bigcup_{i=1}^{\infty} K_i$.$\diamond$

A point $p \in X$ is a *limit point* of a subset A of a topological space X iff each open set in X which contains p also contains an element of A, other than p. An element q of A which is not a limit point of A is called an *isolated point* of A, and then $\{q\}$ is the intersection of A with an open set in X. A subset P of X is called *perfect* iff P is non-empty, closed and it has not isolated points, i.e., each point of P is a limit point of P.

Proposition A.7 (Characterization of perfect subsets of $\mathcal{N}$): If $P \subseteq \mathcal{N}$, the following are equivalent:

[104] Unlike $\mathbb{R}$, which is the union of all $[-n, n]$.

(i) P is perfect.
(ii) P is the body of a splitting tree Π on $\mathbb{N}$.

Proof: ($\Rightarrow$) Since P is closed, $P = [\Pi]$, for some tree Π. If Π is not a splitting tree, $\exists u \in \Pi$ such that:

$$[(\forall w_1, w_2)(u \prec w_1, w_2)] \Rightarrow w_1 \preceq w_2 \ \vee \ w_1 \preceq w_2.$$

But then, $B(u) \cap [\Pi] = \{p\}$, therefore p is isolated, which is absurd.
($\Leftarrow$) If $P = [\Pi]$ for some splitting tree Π, then P is automatically closed. If p is in $B(u) \cap [\Pi]$, then there is always an element $q \neq p$ of P also in $B(u)$, since u always splits into incompatible nodes, therefore to different infinite extensions of it.$\diamond$

Cantor space $\mathcal{C}$ is the body of the Cantor tree $2^{<\mathbb{N}}$, i.e., $\mathcal{C} = 2^{\mathbb{N}}$. Since a Cantor tree is a fan, by Proposition A.6, $\mathcal{C}$ is compact in $\mathcal{N}$. As the proof of Proposition A.6 does not involve the axiom of choice (AC), this proof of the compactness of $\mathcal{C}$ is AC free[105]. The cardinality of $\mathcal{N}$ is c, the cardinality of the continuum, because $\mid \mathcal{N} \mid = \aleph_0^{\aleph_0}$ and since $\mathcal{C} \subset \mathcal{N}$, with $\mid \mathcal{C} \mid = 2^{\aleph_0}$, then,

$$2^{\aleph_0} \leq \aleph_0^{\aleph_0} \leq c^{\aleph_0}.$$

But, $2^{\aleph_0} = c^{\aleph_0} = c$, therefore, $\aleph_0^{\aleph_0} = c$. We reach the same conclusion by showing the equipollence between $\mathcal{N}$ and the set of irrational numbers.

We prove now a topological characterization of $\mathcal{C}$, which was given by Brouwer in [Brouwer 1910], and the construction of the tree involved is repeated, mutatis mutandis, in the topological characterization of $\mathcal{N}$[106].

Since Cantor space is the body of a splitting fan, it is a compact subspace of $\mathcal{N}$, with no isolated points. As a subspace of $\mathcal{N}$ it is a T_1-space, with a countable base of clopen subsets. These properties of $\mathcal{C}$ suffice to characterize Cantor space.

Proposition A.8 (Topological characterization of Cantor space (Brouwer 1910)): A topological space X which is:
(i) T_1,
(ii) compact,
(iii) with a countable base of clopen sets and
(iv) without isolated points,
it is homeomorphic to Cantor space $\mathcal{C}$.

Proof: Let (B_n) a fixed enumeration of the base of clopen sets. If u is a finite sequence of $0, 1$ we define inductively on the length of u a closed (therefore compact) and open subset of X.
$K_{<>} = X$, and if K_u is defined we define $K_{u^\frown 0}$ and $K_{u^\frown 1}$ as follows:
Since X has no isolated points, clopen set K_u is not a singleton. Let $x, y \in K_u$. Since X is T_1, there exists open set V such that, $x \in V$ and $y \notin V$. Hence, there exists B_n, for some n, such that $x \in B_n$ and $B_n \subseteq V$. So, $B_n \cap K_u$ is a proper, non-empty subset of K_u, since $x \in B_n$ and $y \notin B_n$. Let n be the least natural number for which $B_n \cap K_u$ is a proper, non-empty subset of K_u. Therefore, the inductive definition in question is

[105] The AC is used in the proof of Tychonov's theorem, which proves directly the compactness of $\mathcal{C}$.
[106] In the Appendix of this chapter we summon the topological characterizations of all related spaces. We followed [Truss 1999] in the proofs of the following two propositions.

the following:

(i) $K_{<>} = X$,

(ii) $K_{u^\frown 0} = B_n \cap K_u$ and $K_{u^\frown 1} = K_u - B_n$

where B_n is defined from K_u as above.

It is clear that,

(iii) $K_{u^\frown 0} \cap K_{u^\frown 1} = \emptyset$ and

(iv) $K_{u^\frown 0} \cup K_{u^\frown 1} = K_u$.

I.e., $K_{u^\frown 0}$ and $K_{u^\frown 1}$ form a partition of K_u. In that way a tree similar to $\mathcal{C}$ is constructed ($K_{(0)}$ is B_1, if B_1 is not the empty set).

It is obvious that each K_u is formed by the clopen sets B_n, so that it is clopen itself.

If $\alpha \in \mathcal{C}$ to each initial segment n_α of α it corresponds the compact set K_{n_α}, applying the above construction. Because of (iv) it holds that

(v) $K_{1_\alpha} \supset K_{2_\alpha} \supset ... \supset K_{n_\alpha}$,

and since K_{n_α} is by definition non-empty, the family of clopen sets $(K_{n_\alpha})_{n\in\mathbb{N}}$ has the finite intersection property, therefore by the compactness of X,

$$\bigcap_{n\in\mathbb{N}} K_{n_\alpha} \neq \emptyset.$$

Moreover, the above intersection is a singleton i.e.,

$$\bigcap_{n\in\mathbb{N}} K_{n_\alpha} = \{y\}$$

for some $y \in X$. If the intersection contained two elements $y \neq z$, then, by the T_1-condition on X and the existence of the base of all B_n, there is some B_k with $y \in B_k$ and $z \notin B_k$. For each n, $K_{(n+1)_\alpha}$ is, by the inductive definition, equal to $K_{n_\alpha} \cap B_w$ or to $K_{n_\alpha} - B_w$, where $w = w(n)$, and since $w(n)$ are all unequal to each other, then for some n it holds that $w(n) > k$. Hence, since $w(n)$ for that n is the minimum index so that $K_{n_\alpha} \cap B_w$ is a proper, non-empty subset of K_{n_α}, for that index $k < w$ it holds that $K_{n_\alpha} \cap B_k \neq \emptyset$, since both sets contain y, so, necessarily, $(K_{n_\alpha} \cap B_k) = K_{n_\alpha}$ i.e., $B_k \supseteq K_{n_\alpha}$. But, if z belongs to the infinite intersection, then it belongs also to K_{n_α} i.e., to B_k, which is absurd, by the definition of B_k.

From the above the following map $\theta : \mathcal{C} \to X$ is well-defined, where

$$\theta(\alpha) = y \leftrightarrow \bigcap_{n\in\omega} K_{n_\alpha} = \{y\}.$$

We prove that θ is a homeomorphism.

θ is $1-1$ map: If $\alpha \neq \beta$ elements of $\mathcal{C}$, then α, β differ in some initial segment, and suppose that this happens for the fist time in the $(n+1)$-level. If $\theta(\alpha) = y$ and $\theta(\beta) = z$, then $y \in K_{n_\alpha{}^\frown 0}$ and $z \in K_{n_\alpha{}^\frown 1}$, which are disjoint, therefore $y \neq z$.

θ is onto X: If $y \in X$, then using the Principle of the Excluded Middle (PEM), we find $\alpha \in \mathcal{C}$, such that $\theta(\alpha) = y$, as follows:

In each step of the tree of K_u, y belongs to one of the two sets i.e., $y \in K_{(0)}$ or $y \in K_{(1)}$ and if, for example, $y \in K_{(0)}$, then $y \in K_{(0,0)}$ or $y \in K_{(0,1)}$ etc. By that way it is evident how the element of $\mathcal{C}$ which corresponds to y through θ is constructed.

θ corresponds a basic clopen set of $\mathcal{C}$ to a clopen set of X:

$$\theta(B_k(\alpha)) = K_{k_\alpha}.$$

$\theta(B_k(\alpha)) \subseteq K_{k_\alpha}$, since $\theta(\alpha) = y \leftrightarrow \bigcap_{n\in\mathbb{N}} K_{n_\alpha} = \{y\}$, so $y \in K_{k_\alpha}$.
Also, $\theta(B_k(\alpha)) \supseteq K_{k_\alpha}$, since, if $y \in K_{k_\alpha}$, then $y \in K_{k_{\alpha^\frown 0}}$ or $y \in K_{k_{\alpha^\frown 1}}$, so, by that way (as in the onto case) an element of $B_k(\alpha)$ is formed, which corresponds through θ to y.
A basic clopen set $_w$ of X has as a θ-inverse image a clopen set of $\mathcal{C}$:
$\theta^{-1}(B_w)$ is a closed set of $\mathcal{C}$:
Let α^n a sequence in $\mathcal{C}$ such that $\alpha^n \to \alpha$ (i.e., all terms of it with index larger than some index n_0 have the same m-initial segment with α) and if for each n, $\theta(\alpha^n) \in B_w$, where if $\theta(\alpha^n) = y^n \leftrightarrow \bigcap_{k\in\mathbb{N}} K_{k_{\alpha^n}} = \{y^n\}$, then $y^n \in B_w$. We show that $\theta(\alpha) = y \leftrightarrow \bigcap_{m\in\mathbb{N}} K_{m_\alpha} = \{y\}$ belongs to B_w too. Since m_α is the m-initial segment of all α^n, for $n \geq n(m)$, then

$$(K_{m_\alpha} \cap B_w) \neq \emptyset,$$

since $y^n \in K_{m_\alpha} \cap B_w$, for $n \geq n(m)$. Since B_w is a closed subset of a compact space, it is compact itself, and the family of closed sets $(K_{m_\alpha} \cap B_w)_m$ has the finite intersection property (a finite intersection of elements of this family is equal to the non-empty intersection of the smallest set of the finite subfamily - the one, due to (v), with the largest index - with B_w). Hence, by the compactness of B_w, the whole family has non-empty intersection, which is equal to $\{y\} \cap B_w$, which shows that $y \in B_w$.
$\theta^{-1}(B_w)$ is an open set: It suffices to show that the complement of $\theta^{-1}(B_w)$ in $\mathcal{C}$ is closed. If α^n is a sequence in $\mathcal{C}$ such that $\alpha^n \to \alpha$ and $\theta(\alpha^n) \notin B_w$ for each n, it can be proved, as above, that $\theta(\alpha) \notin B_w$.$\diamond$.

The hypothesis that X has no isolated points was crucial to the definition of K_u, so that with the compactness of X, $\bigcap_{n\in\mathbb{N}} K_{n_\alpha} \neq \emptyset$. Without these two hypotheses we have the following generalization.

Proposition A.9: If X is a T_1 topological space with a countable base of clopen sets, then X is homeomorphic to a subspace of $\mathcal{C}$.

Proof: If $(B_n)_n$ a base of clopen sets of X, then we define the following tree of clopen sets C_u:

$$C_{<>} = X \quad C_{u^\frown 0} = C_u \cap B_n \quad C_{u^\frown 1} = C_u - B_n.$$

If $\alpha \in \mathcal{C}$, then, as in the above proof, $\bigcap_{n\in\mathbb{N}} C_{n_\alpha}$ has at most one element. Let Γ the set of all α in $\mathcal{C}$ for which the above intersection is non-empty. We define again a map $\theta : \Gamma \to X$ by $\theta(\alpha) = y \leftrightarrow \bigcap_{n\in\mathbb{N}} C_{n_\alpha} = \{y\}$. The fact that θ is a homeomorphism is proved as in the previous proof.$\diamond$

Corollary: If $n \in \mathbb{N}$, then $n^{\mathbb{N}}$ is homeomorphic to $\mathcal{C}$.
Clearly, $n^{\mathbb{N}}$ satisfies all hypotheses of the characterization of $\mathcal{C}$.

Brouwer's characterization of $\mathcal{C}$ is extended to Cantor cubes of larger weight i.e., to spaces 2^λ, where λ is an uncountable cardinal.

As we have already shown, $\mathcal{N}$ has a base of clopen sets, the only clopen and compact subset of $\mathcal{N}$ is the empty set, and it is also complete. These properties suffice to characterize $\mathcal{N}$.

Proposition A.10 (Topological characterization of Baire space (Alexandroff,

Urysohn 1928)): A topological space X which is:
(i) completely metrizable,
(ii) with a countable base of clopen subsets,
(iii) without non-empty, clopen and compact subset,
it is homeomorphic to Baire space $\mathcal{N}$.

Proof: Let (X, d) the complete metric space of our hypothesis and $(B_n)_n$ a fixed enumeration of a countable base of clopen sets. If someone tries to mimic Brouwer's construction needs something like the following lemma.

Lemma 1: If (X, d) a metric space satisfying the above hypotheses and $(B_n)_n$ its fixed base of clopen sets, then each non-empty, clopen subset of X is written as

$$H = \bigcup_{n \in S} H_n,$$

where
(a) S is countable,
(b) Each H_n is non-empty, clopen,
(c) $H_n \cup H_m = \emptyset$,
(d) $diam(H_n) \leq \delta$, where $diam(H_n)$ is the diameter of H_n and δ is any positive number.

Proof of Lemma 1: For its proof we need the following lemma.

Lemma 2: If $(B_n)_n$ is a countable base of a metric space (X, d), then the family of B_n with diameter $\leq \delta$ is also a base for the topology of X.

Proof of Lemma 2: Let A be open and $x \in A$. We shall find an element of the above family containing x and contained in A.
Let $A_x^{'} = B(x, \frac{\delta}{2}) \cap A = \{y \in A : d(x, y) < \frac{\delta}{2}\}$, which is open, as the intersection of open sets, and it has diameter $\leq \delta$ (if $y, z \in A_x^{'}$, then $d(y, z) \leq d(y, x) + d(x, z) < \delta$). Since $x \in A_x^{'}$, there is a basic set B_n containing x, contained in $A_x^{'}$, and, therefore, it has also diameter $\leq \delta$.

Going back to the proof of Lemma 1, we see that the clopen set H cannot be, by hypothesis, compact, and so each cover of H does not have necessarily a finite subcover. So, if $H \subseteq \bigcup_{i \in I} A_i$, then A_i's, as intersections of open sets in X with the open H, are open in X too.
If we define the sets $S_i = \{n \in \mathbb{N} : B_n \subseteq A_i \ \wedge \ diam(B_n) \leq \delta\}$, then, by Lemma 1, $A_i = \bigcup_{n \in S_i} B_n$ and if we define $S = \bigcup_{i \in I} S_i$ we find that

$$H = \bigcup_{n \in S} B_n.$$

Of course, this cover of H has no finite subcover, otherwise its initial cover would also have one, which is absurd.
In the standard way, we "disjoint" the family of B_n with $n \in S$. I.e., we define for each $n \in S$ the set $H_n = B_n - \bigcup_{k<n, k \in S} B_k$. Sets H_n are by their definition clopen sets of X, mutually disjoint, with diameter $\leq \delta$, and they satisfy the following property:

$$\bigcup_{m \leq n, m \in S} H_m = \bigcup_{m \leq n, m \in S} B_m.$$

Hence, since all B_n, $n \in S$ cover H, all H_n, $n \in S$ cover H too. Moreover, the set $\{n \in S : H_n \neq \infty\}$ is a countable infinite one, otherwise, by the above equality, there would be a finite subcover from B_n, $n \in S$, of H.

Going back to the proof of the main theorem, we construct, as in the proof of the characterization of $\mathcal{C}$, a tree of clopen sets of X, which is going to be a "copy" of $\mathcal{N}$. We correspond inductively to each finite sequence u of naturals a clopen set of X as follows:

(i) $K_{<>} = X$,
(ii) K_u splits into $K_{u^\frown n}$, where $n \in \mathbb{N}$ and the family of $K_{u^\frown n}$ is that partition of K_u by clopen sets with diameter $\leq \frac{1}{l(u)}$, the existence of which is established by Lemma 1.

In analogy to Brouwer's tree, if $\alpha \in \mathcal{N}$, then

$$K_{1_\alpha} \supset K_{2_\alpha} \supset \ldots \supset K_{n_\alpha},$$

and

$$\bigcap_{n \in \mathbb{N}} K_{n_\alpha} \neq \emptyset$$

since, for each n, let $x_n \in K_{n_\alpha}$ (principle of dependent choices), since $K_{n_\alpha} \neq \emptyset$. Then, if $m \leq n$, then $x_m, x_n \in K_{m_\alpha}$, because $K_{n_\alpha} \subset K_{m_\alpha}$. Hence, by the definition of K_{m_α}, $d(x_m, x_n) \leq \frac{1}{m}$. So, for an as small as possible positive $\frac{1}{m}$ there is a natural (m itself) such that, for $n \geq m$, $d(x_m, x_n) \leq \frac{1}{m}$. I.e., the sequence $(x_n)_n$ is a Cauchy sequence. Therefore, by completeness of X, there is $x \in X$ such that $x_n \to x$.
But, $x \in \bigcap_{n \in \mathbb{N}} K_{n_\alpha}$ since, if for some n, $x \notin \bigcap_{n \in \mathbb{N}} K_{n_\alpha}$, then x has a strictly positive distance from the clopen set K_{n_α}. By the convergence of $(x_n)_n$ to x though, there are terms of the sequence which belong to K_{n_α} and their distance from x is less than this positive distance, something which is absurd.
Moreover, the above intersection is a singleton. I.e.,

$$\bigcap_{n \in \mathbb{N}} K_{n_\alpha} = \{y\}$$

for some $y \in X$. If there was another element in the intersection, it would have some strictly positive distance from y and at the same time they would both belong to the K_{n_α} sets of diameter less than this positive distance[107].
Again we define a map $\theta : \mathcal{N} \to X$ by,

$$\theta(\alpha) = y \leftrightarrow \bigcap_{n \in \mathbb{N}} K_{n_\alpha} = \{y\}.$$

θ is proved to be $1-1$, onto X and a homeomorphism exactly like the $\mathcal{C}$-case. I.e., again $\theta(B_k(\alpha)) = K_{k_\alpha}$ and $\theta^{-1}(K_u)$ is clopen in $\mathcal{N}$.
Therefore, if we show that sets K_u, where $u \in \mathbb{N}^{\mathbb{N}}$ form a basis for the topology of X, then θ sends all basic clopen sets of $\mathcal{N}$ to basic sets of X and the inverse image of basic sets of X are clopen sets of $\mathcal{N}$.

[107] We reach the same conclusion using the fact that a metric space X is complete iff for each descending sequence $(F_n)_n$ of non-empty closed subsets of X, with $\lim diam(F_n) = 0$, there exists an x such that $\bigcap_{n \in \mathbb{N}} F_n = \{x\}$.

Let A open in X and $y \in A$. There is then $B(y, \frac{1}{n}) \subseteq A$. Let α in $\mathcal{N}$ such that $\theta(\alpha) = y \leftrightarrow \bigcap_{n\in\mathbb{N}} K_{n_\alpha} = \{y\}$. Then, $K_{n+1_\alpha} \subseteq A$, because, since $y \in K_{n+1_\alpha}$, if $z \in K_{n+1_\alpha}$, then $d(z,y) \leq \frac{1}{n+1} < \frac{1}{n}$, hence $z \in B(y, \frac{1}{n})$.
Then, the homeomorphism between $\mathcal{N}$ and X has been established.$\diamond$

As a corollary of the above proof we get, in analogy to Proposition A.9, the following fact.

Proposition A.11: If X is a topological space with a countable base of clopen sets and without a non-empty, clopen and compact subset, then X is embedded to $\mathcal{N}$.

As we mentioned at the beginning of the Appendix, real spaces are only "slightly" different. We have already stated all necessary tools to show the exact relations between real spaces.
Cantor set Ca is generated by $[0,1]$ by pulling out its middle third interval $(\frac{1}{3}, \frac{2}{3})$, and then, by pulling out the middle third interval of the remaining two intervals, and so on. The remaining set after the absolutely infinite completion of this procedure is Ca. It is easy to see that a point of $[0,1]$ belongs to Ca iff its triadic expansion contains no 1. I.e, an element c of Ca is of the form

$$c = \sum_{i=1}^{\infty} \frac{c_i}{3^i} \quad , \quad c_i \in \{0, 2\}.$$

Proposition A.12: Cantor set Ca is homeomorphic to Cantor space $\mathcal{C}$.

Proof: The form of an element c of Ca shows that Ca is closed in $[0,1]$, therefore it is compact. Also, it has no isolated points and it has a countable base of clopen sets, since it contains no non-trivial intervals (a non-empty subset of $\mathbb{R}$ has a base of clopen sets iff it contains only trivial intervals). Therefore, by Proposition A.8, it is homeomorphic to $\mathcal{C}$[108].$\diamond$

Proposition A.13: A separable, zero-dimensional metric space X is embedded to $\mathbb{R}$.

Proof: X has a countable base of clopen sets, therefore, by Proposition A.9, it is embedded to $\mathcal{C}$. Since $\mathcal{C}$ is homeomorphic to Ca, a subset of $[0,1]$, X is embedded to $\mathbb{R}$.$\diamond$

As in the case of $\mathcal{C}$, there are topological characterizations of all spaces of the form $\mathbb{N}^\lambda$, where λ is an uncountable cardinal[109].

Proposition A.14: The set of irrational numbers $\mathcal{I}$ is homeomorphic to $\mathcal{N}$.

Proof: The intervals (closed or open) with rational ends cutting $\mathcal{I}$ form a countable base of clopen sets in $\mathcal{I}$.
Also $\mathcal{I}$ with the relative topology of $\mathbb{R}$ has no non-empty, compact and clopen subset. If K was such a set, then it would be the intersection of an open in $\mathbb{R}$ set with $\mathcal{I}$. Hence, there would be a basic set $B(x,\varepsilon)$ such that, $B(x,\varepsilon) \cap \mathcal{I} \subseteq K$. In the intersection there are Cauchy sequences of rationals converging to rational points outside K, which is absurd, since K is complete, as a compact metric space.
Finally, although $\mathcal{I}$ is not complete with the relative topology of $\mathbb{R}$, it is completely metrizable with a complete metric equivalent to its natural metric. We use here a the-

[108] The standard homeomorphism between $\mathcal{C}$ and Ca is a special case ($X = \mathcal{C}$) of the homeomorphism in the proof of Proposition A.8.
[109] See [Chigogidze 1996].

orem of Mazurkiewisz, according to which, a subset of a complete metric space X is G_δ iff it is completely metrizable with a complete metric equivalent to the initial relative metric. But $\mathcal{I}$ is G_δ, as a complement of $\mathcal{Q}$, which is F_σ[110].
Therefore, $\mathcal{I}$ satisfies the hypotheses of the characterization of $\mathcal{N}$[111].$\diamond$

We may establish the aforementioned homeomorphism independently from the characterization of $\mathcal{N}$. Here we shall only sketch this proof[112].
$\mathcal{N}$ is identified with its subset A of sequences containing no 0:

$$(\alpha_0, \alpha_1, ..., \alpha_n, ...) \mapsto (\alpha_0 + 1, \alpha_1 + 1, ..., \alpha_n + 1, ...).$$

Then, we define a map $\theta : A \to (1, +\infty)$ as follows:

$$\theta(\alpha_0, \alpha_1, ..., \alpha_n, ...) = \alpha_0 + \cfrac{1}{\alpha_1 + \frac{1}{\alpha_2 + \frac{1}{\alpha_3 + \dots}}}.$$

Using basic properties of continued fractions[113] it is proven that:

(i) $\theta(\alpha_0, \alpha_1, ..., \alpha_n, ...)$ converges.
(ii) θ is $1-1$.
(iii) $\theta(\alpha_0, \alpha_1, ..., \alpha_n, ...)$ is irrational.
(iv) Each irrational > 1 is in the range of θ.
(v) θ and θ^{-1} are continuous.

Finally, through a homeomorphism $f : (1, +\infty) \to \mathbb{R}$

$$f(x) = \begin{cases} x - 1 & \text{, if } x \geq 2 \\ 2 - \frac{1}{x-1} & \text{, if } 1 < x < 2 \end{cases}$$

which preserves irrationality, we get, composing all previous homeomorphisms, a homeomorphism between $\mathcal{N}$ and $\mathcal{I}$.

A (classical) function $f : X \to Y$, where (X, ρ_1) and (Y, ρ_2) are metric spaces, is continuous iff the inverse image of an open set in Y through f is an open set in X. Equivalently, if for each $\alpha \in X$

$$(1) \quad \forall \varepsilon > 0 \ \ \exists \delta > 0 : \ \ \rho_1(\beta, \alpha) < \delta \ \ \Rightarrow \ \ \rho_2(f(\beta), f(\alpha)) < \varepsilon.$$

Equivalent to (1) is the following:

$$(2) \quad \forall \lambda \in \mathbb{N}, \ \ \exists k \in \mathbb{N} : \ \ \rho_1(\beta, \alpha) < \frac{1}{k} \ \ \Rightarrow \ \ \rho_2(f(\beta), f(\alpha)) < \frac{1}{\lambda}.$$

If $f : \mathcal{N} \to \mathbb{N}$, where $\mathbb{N}$ is a discrete metric space, (2) has the following form:

$$(3) \quad \exists k \in \mathbb{N} : \ \beta, \alpha \ \ k\text{-same} \Rightarrow f(\beta) = f(\alpha).$$

[110] $\mathcal{Q}$ is F_σ, since it is the countable union of the closed singletons of rationals.
[111] By this proof it is evident why completeness of metric spaces is not a topological invariant.
[112] A complete proof can be found in [Truss 1997] Chapter 10.
[113] A classical introduction to continued fractions is [Khinchin 1964], while an original interpretation of Plato's dialectics through the concept of anthyphairesis, a concept equivalent to that of a continued fraction, can be found in Negrepontis work (see e.g., [Negrepontis 2005]).

(3) expresses the fact that a map $f : \mathcal{N} \to \mathbb{N}$ is continuous if its value $f(\alpha)$ is determined by some initial segment of α. Of course, not every map $f : \mathcal{N} \to \mathbb{N}$ is continuous. E.g.,

$$f(\alpha) = \begin{cases} 0 & , \text{if } \alpha = \overline{0} \\ 1 & , \text{if } \alpha \neq \overline{0} \end{cases}$$

Obviously, there is no initial segment of $\overline{0}$ which determines the value $f(\overline{0})$, since for each $N_{\overline{0}}$, there is a sequence $\beta \neq \overline{0}$ such that $N_\beta = N_{\overline{0}}$ and it has value 1 under f.
If $f : \mathcal{N} \to \mathcal{N}$, then continuity condition (2) becomes:

$$(4) \quad \forall \lambda \in \mathbb{N}, \ \ \exists k \in \mathbb{N} : \ \ \rho(\beta, \alpha) < \frac{1}{k} \ \ \Rightarrow \ \ \rho(f(\beta), f(\alpha)) < \frac{1}{\lambda}.$$

I.e.,

$$(5) \quad \forall \lambda \in \mathbb{N}, \ \ \exists k \in \mathbb{N} : \ \ \beta, \alpha \ k\text{-same} \Rightarrow f(\beta), f(\alpha) \ \lambda\text{-same}.$$

Another characterization of continuity is through sequential continuity i.e., f is continuous iff for each $\alpha \in \mathcal{N}$, and (β_n) a sequence in $\mathcal{N}$,

$$(6) \quad \beta_n \to \alpha \Rightarrow f(\beta_n) \to f(\alpha).$$

Proposition A.15: The following are equivalent:
(i) $f : \mathcal{N} \to \mathcal{N}$ is continuous.
(ii) There is a map $f^* : \mathbb{N}^{<\infty} \to \mathbb{N}^{<\infty}$, such that:
(a) If $N \leq M$, then $f^*(\alpha_1, \alpha_2, ..., \alpha_N) \preceq f^*(\alpha_1, \alpha_2, ..., \alpha_M)$, i.e., f^* is monotone.
(b) f^* is not stagnant.
(c) f on α is given by

$$f(\alpha) = \sup_N f^*(N_\alpha),$$

i.e., sequence $f(\alpha)$ is approximated by finite sequences $f^*(N_\alpha)$.

Proof: This is a direct consequence of Proposition 2.2, if $X = Y = \mathbb{N}$ and $T = S = \mathbb{N}^{<\infty}$. Obviously f^* is proper, because of (b).$\diamond$

Of course, not every map $f : \mathcal{N} \to \mathcal{N}$ is continuous. E.g.,

$$f(\alpha) = \begin{cases} \overline{0} & , \text{if } \alpha = \overline{0} \\ \overline{1} & , \text{if } \alpha \neq \overline{0} \end{cases}$$

Consider the sequence (β_n) of elements of $\mathcal{N}$, such that

$$\beta_n = (\underbrace{0, 0, , ..., 0}_{n}, \overline{1}).$$

Then, $\beta_n \to \overline{0}$, since $\beta_n \in B(\overline{0}, \frac{1}{n})$, but $f(\beta_n) \to \overline{1}$, while $f(\overline{0}) = \overline{0}$.
Next result shows how easy it is, due to Proposition A.15 $(ii) \Rightarrow (i)$, to find that a certain map $\mathcal{N} \to \mathcal{N}$ is continuous.
We define the following symbol:

$$2_n^k = \begin{cases} (\underbrace{1, 1, ..., 1}_{n}, 0) & , \text{if } k = \text{even} \\ (\underbrace{0, 0, ..., 0}_{n}, 1) & , \text{if } k = \text{odd} \end{cases}$$

Obviously,

$$2^k_0 = \begin{cases} (0) & \text{, if } k = \text{even} \\ (1) & \text{, if } k = \text{odd} \end{cases}$$

We define the **standard injection Θ between $\mathcal{N}$ and** $\mathcal{C}$, $\Theta : \mathcal{N} \to \mathcal{C}$, which is also a $\mathcal{N} \to \mathcal{N}$ map, as follows:

$$\Theta(\alpha_1, \alpha_2, ..., \alpha_n, ...) = 2^1_{\alpha_1} \frown 2^2_{\alpha_2} \frown ... \frown 2^n_{\alpha_n} \frown ...,$$

where $2^2_{\alpha_2} \frown ... \frown 2^n_{\alpha_n} \frown ...$ denotes the concatenation of all finite sequences $2^i_{\alpha_i}$. Hence,

$$\Theta(\alpha_1, \alpha_2, ..., \alpha_n, ...) = (\underbrace{0, 0, ..., 0}_{\alpha_1}, 1, \underbrace{1, 1, ..., 1}_{\alpha_2}, 0, \underbrace{0, 0, ..., 0}_{\alpha_3}, 1...).$$

Obviously, the value of a sequence α under Θ would be slightly different if we had considered $\Theta(\alpha_0, \alpha_1, ..., \alpha_n, ...) = 2^0_{\alpha_0} \frown 2^1_{\alpha_1} \frown ... \frown 2^n_{\alpha_n} \frown ...$[114].

Proposition A.16: The standard injection $\Theta : \mathcal{N} \to \mathcal{C}$ is a homeomorphism of the Baire space on the subspace of all non-stagnant elements of Cantor space.

Proof: $\Theta^* : \mathcal{N}^{<\infty} \to 2^{<\infty}$ is defined like Θ and it is obviously monotone, non-stagnant (i.e., proper) and trivially $\Theta(\alpha) = \sup_N \Theta^*(N_\alpha)$. So, Θ is continuous. It is clearly $1-1$ and its inverse is easy to define. E.g., $\Theta^{-1}(0, 0, 0, 1, 0, 1, ...) = (3, 0, 0, ...)$. The gradual construction of $\Theta^{-1}(\beta_1, \beta_2, ..., \beta_n, ...)$, where $(\beta_1, \beta_2, ..., \beta_n, ...)$ is a non-stagnant sequence of $\mathcal{C}$, shows that Θ^{-1} is also continuous. By its definition, $\Theta(\alpha)$ cannot be a stagnant sequence of $\mathcal{C}$. Then, Θ is a homeomorphism of $\mathcal{N}$ on the non-stagnant elements of $\mathcal{C}$.$\diamond$

The above results provides an example of a closed set of $\mathcal{N}$, the continuous image of which is not closed in $\mathcal{N}$.
$\mathcal{N}$ is trivially closed and $\Theta(\mathcal{N})$ is not, since

$$\alpha^n = (\underbrace{1, 1, ..., 1}_{n}, non-stagnant\ tail)$$

belongs to $B(\overline{1}, \frac{1}{n}) \cap \Theta(\mathcal{N})$, therefore, $\alpha^n \to \overline{1}$, but $\overline{1} \notin \Theta(\mathcal{N})$.
If $f : \mathcal{N} \to \mathbb{N}$, then the general uniform continuity condition

$$(\exists N)(\forall \varepsilon > 0) \ \ d(\alpha, \beta) < \frac{1}{N} \Rightarrow d(f(\alpha), f(\beta)) < \varepsilon$$

becomes

$$(\exists N)(\forall \alpha, \beta) \ \ N_\alpha = N_\beta \Rightarrow f(\alpha) = f(\beta),$$

[114] If we had defined Θ by

$$\Theta(\alpha_1, \alpha_2, ..., \alpha_n, ...) = 2(\alpha_1) \frown 2(\alpha_2) \frown ... \frown 2(\alpha_n) \frown ...,$$

where

$$2(n) = \begin{cases} (\underbrace{1, 1, ..., 1}_{n}) & \text{, if } n = \text{even} \\ (\underbrace{0, 0, ..., 0}_{n}) & \text{, if } n = \text{odd} \end{cases}$$

then 1 is not the value of some n, therefore the definition of Θ cannot be completed.

just by taking $\varepsilon < 1$.

Proposition A.17 ($\mathcal{C}$ and $\mathcal{N}$ are dimensionless): $\mathcal{C}^j$ is homeomorphic to $\mathcal{C}$ and $\mathcal{N}^j$ is homeomorphic to $\mathcal{N}$, where $j \in \mathbb{N}$ or $j = \mathbb{N}$.

Proof: We prove that $\mathcal{C}^{\mathbb{N}}$ is homeomorphic to $\mathcal{C}$ and $\mathcal{N}^{\mathbb{N}}$ is homeomorphic to $\mathcal{N}$, since the case $j \in \mathbb{N}$ is proved then in an obviously similar manner.
The first proof we give is through the topological characterizations of $\mathcal{C}$ and $\mathcal{N}$.
It suffices to show that $\mathcal{C}^{\mathbb{N}}$ is T_1, compact, with a countable base of clopen sets and with no isolated points.
$\mathcal{C}^{\mathbb{N}}$ has the product topology and since $\mathcal{C}$ is T_1, $\mathcal{C}^{\mathbb{N}}$ is also T_1, and compact, by Tychonoff's theorem. A base for its topology is the collection

$$\{\prod_{n\in\mathbb{N}} B_n |\ B_n \in \mathfrak{B} \ or \ B_n = \mathcal{C} \ and \ |\{n|B_n \neq \mathcal{C}\}| < \infty\},$$

where $\mathfrak{B}$ denotes the countable base of clopen sets of $\mathcal{C}$. $\mathfrak{B}$ is countable, therefore, the above base is also countable. Since

$$(\prod_{n\in\mathbb{N}} B_n)^- = \prod_{n\in\mathbb{N}} B_n^- = \prod_{n\in\mathbb{N}} B_n,$$

$\prod_{n\in\mathbb{N}} B_n$ are closed and since

$$(\prod_{n\in\mathbb{N}} B_n)^\circ \subseteq \prod_{n\in\mathbb{N}} B_n^\circ = \prod_{n\in\mathbb{N}} B_n,$$

and the inverse

$$\prod_{n\in\mathbb{N}} B_n^\circ \subseteq (\prod_{n\in\mathbb{N}} B_n)^\circ$$

also holds, because there is by the definition of the product base a finite set J of $\mathbb{N}$ for which $B_j = \mathcal{C}$, for each $j \in \mathbb{N} - J$, then $\prod_{n\in\mathbb{N}} B_n$ are also open.
$\mathcal{C}^{\mathbb{N}}$ has no isolated points because if $\{\alpha^1, \alpha^2, ..., \alpha^m, ...,\}$ was open there is no basic set $\prod_{n\in\mathbb{N}} B_n$ contained in it, since most of the components of $\prod_{n\in\mathbb{N}} B_n$ are $\mathcal{C}$ itself.
In a similar way to the case of $\mathcal{N}$ it suffices to show that $\mathcal{N}^{\mathbb{N}}$ has a countable base of clopen sets, it has no non-empty clopen and compact subsets and it is metrizable as a complete metric space.
The existence of a countable base of clopen sets is derived as in the previous case. If K was a non-empty clopen and compact subset of $\mathcal{N}^{\mathbb{N}}$, then $\pi_n(K)$, where π_n is the n-projection mapping, is also non-empty, compact (π_n is continuous), and clopen (π_n is open mapping and $\pi_n(K)$ is closed, since it is compact) subset of $\mathcal{N}$, which is absurd.
Also, it is standard that if (X_n, ρ_n) is a sequence of complete metric spaces, such that $\rho_n(x, y) \leq 1$, then the product $X = \prod_{n\in\mathbb{N}} X_n$ with the product metric is complete metric space. If $X_n = \mathcal{C}$ and $\rho_n = \rho$, we get the last property of $\mathcal{N}$'s characterization for $\mathcal{N}^{\mathbb{N}}$.

We can also prove the same results in a more direct way. We describe how for the $\mathcal{N}$-case, but it works for the $\mathcal{C}$-case too.

Let P is a partition of $\mathbb{N}$, such that all elements of P are infinite too[115]. I.e.,

$$\mathbb{N} = \bigcup_{i \in I} N_i,$$

where $|I| = \aleph_0$, $N_i \cap N_j = \emptyset$ and $|N_i| = \aleph_0$, for each i, j. Let

$$N_i = \{k_1^i, k_2^i, , ..., k_n^i, ..., \}.$$

We define $e : \mathcal{N} \to \mathcal{N}^{\mathbb{N}}$ as follows:

$$e(\alpha) = (\alpha^1, \alpha^2, ..., \alpha^m, ...,)$$

where,

$$\alpha^m(n) = \alpha(k_n^m).$$

e is $1-1$, since if $\alpha \neq \beta$, then there is i such that $\alpha(i) \neq \beta(i)$, where $i = k_n^m$ for some m, n. But then, $e(\alpha) \neq e(\beta)$ since otherwise, $\alpha^m = \beta^m$, therefore, $\alpha^m(n) = \beta^m(n)$, i.e., $\alpha(i) = \beta(i)$, which is absurd.
e is onto $\mathcal{N}^{\mathbb{N}}$, since for each element $(\alpha^1, \alpha^2, ..., \alpha^m, ...,)$ of $\mathcal{N}^{\mathbb{N}}$, we define α by $\alpha(i) = \alpha(k_n^m) = \alpha^m(n)$ and, obviously, $e(\alpha) = (\alpha^1, \alpha^2, ..., \alpha^m, ...,)$.
e is sequentially continuous, therefore continuous. We know that

$$\alpha^j \to \alpha \Leftrightarrow \forall \lambda \ \exists m(\lambda) \ \forall m \ m > m(\lambda) \ \ \alpha^m(\lambda) = \alpha(\lambda).$$

If

$$e(\alpha^j) = (\alpha^{j^1}, \alpha^{j^2}, ..., \alpha^{j^m}, ...,),$$

then we know that $e(\alpha^j) \to e(\alpha)$ iff $\alpha^{j^i} \to \alpha^i$, for each i.
The last condition is justified as follows: If we fix a natural λ, then

$$\alpha^{j^i}(\lambda) = \alpha^j(k_\lambda^i) = \alpha(k_\lambda^i),$$

for $j > m(k_\lambda^i)$, and since $\alpha(k_\lambda^i) = \alpha^i(\lambda)$, the above convergence is established.
The continuity of e^{-1} is proved likewise.$\diamond$

We add, in summary, the following topological characterizations of related spaces.

1. Topological characterization of the unit interval (Veblen 1905): Every connected and locally connected metric space with two non-cut points[116] is homeomorphic to the unit interval $[0, 1]$.

2. (Frechet 1910): A countable metrizable space is homeomorphic to a subset of rational numbers.

3. Topological characterization of rationals (Sierpinski 1920):
(a) A countable dense to itself metric space is homeomorphic to the space of rational numbers.

[115] We can construct such a partition, for example, as follows: N_1 is the first natural, the third natural, etc., i.e., the odd numbers. N_2 is the first, the third, etc., of the remaining ones, and so on. Each even number eventually is contained in some N_i for some i.

[116] If X is a connected topological space and $x \in X$, then x is a *non-cut point* iff $X - \{x\}$ is connected, otherwise it is called a *cut point*. Obviously all points of $[0, 1]$ other than 1 or 2 are cut points.

(b) A countable T_1 topological space with a countable base of clopen sets and with no isolated points is homeomorphic to the space of rational numbers[117].

4. (Mazurkiewicz): A G_δ set which is itself and its complement dense in a separable, zero-dimensional and metrizable space, is homeomorphic to the space of irrational numbers.

5. Topological characterization of reals
(a) (Ward 1936): A separable, connected and locally connected metric space X, for each element x of which $X - \{x\}$ has exactly two connected components, is homeomorphic to the space of real numbers.

(b) (Franklin, Krishnarao 1970): The above also holds if the hypothesis of metric space is replaced by that of a regular topological space.

6. (Gruenhage, Schoenfeld 1975): A compact metric space which has, up to homeomorphism, only two non-empty open subsets, is homeomorphic to the Cantor set.

7. Topological characterization of Cantor space (Brouwer 1910): A T_1, compact, with a countable base of clopen sets and with no isolated points topological space is homeomorphic to Cantor space.

8. Topological characterization of Baire space (Alexandroff, Urysohn 1928): A topological space with a countable base of clopen sets, without non-empty clopen and compact subsets, which is also completely metrizable space is homeomorphic to Baire space.

[117] A proof of it can be found e.g., in [Truss 1997] pp.254-5.

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